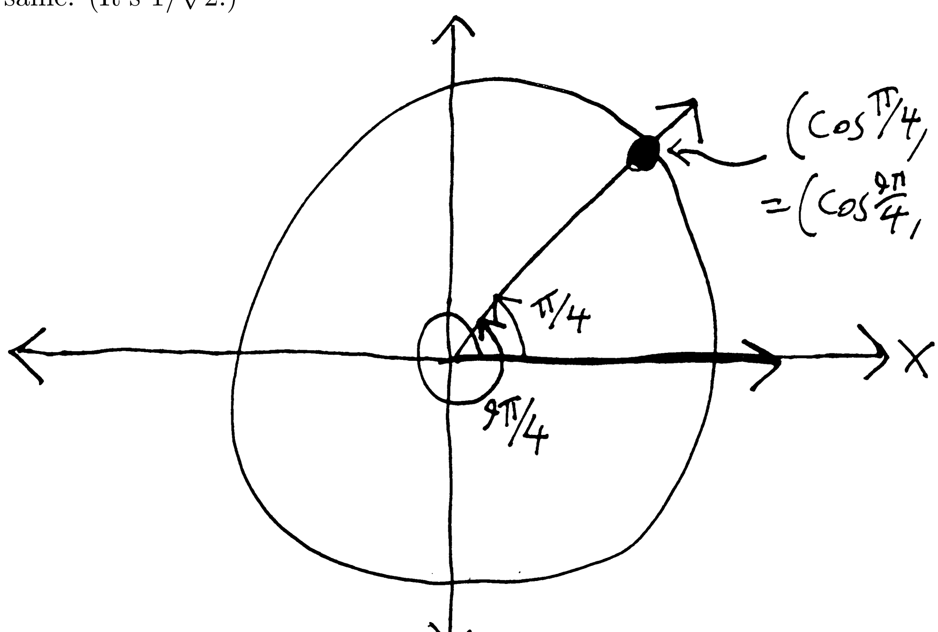
Trig Identities

Pre/Calculus 11, Veritas Prep.

Periodicity Identities

What if we're trying to find the sine of $9\pi/4$? We know this will be exactly the same as $\sin(\pi/4)$. Why? Because $9\pi/4$ is just the same as $\pi/4 + 2\pi$, and the extra 2π is just another revolution around the unit circle—it's a different angle than $\pi/4$, but it lands you back in the same place that $\pi/4$ does. So the sine is the same! (It's $1/\sqrt{2}$.)



This is, not surprisingly, a general law. Both sine and cosine repeat every time we go around the unit circle. Put differently, every time we add 2π to an angle, the sine and cosine don't change. And we could add 2π as many times as we want—we could add 4π, 6π, 358π—or we could even go around the circle in the opposite direction, and subtract 2π, or 4π, or 358π. More formally, the following equations (in which k is any integer) are true:

$$\sin(\theta + 2k\pi) = \sin(\theta)$$

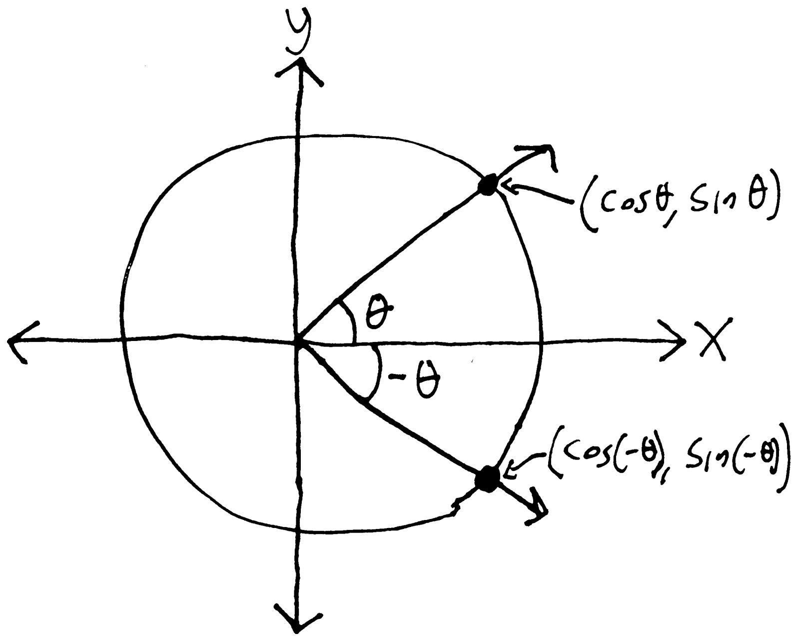
$$\cos(\theta + 2k\pi) = \cos(\theta)$$

What about tangent? It repeats even faster than sine and cosine. Since the only thing that changes about sine and cosine every π units is its sign (s-i-g-n, i.e., whether it's $+$ or $-$), and since tangent is just the quotient of sine and cosine, the signs will cancel out. And so tangent repeats every π units. Formally:

$$\tan(\theta + k\pi) = \tan(\theta)$$

Symmetry Identities

There are some other useful identities. What if I have some angle θ, and I find $\cos(\theta)$, and then I also want to find $\cos(-\theta)$? Is there any relationship between these two cosines? Think about what this will look like on a unit circle:



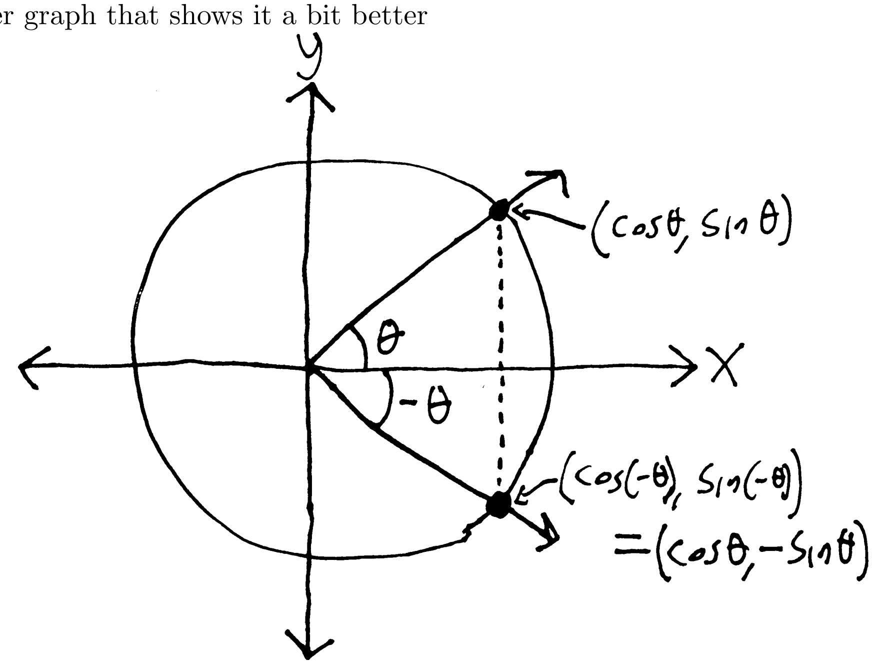
Cosine, of course, is just the x-coordinate of that point on the unit circle. So if we consider $-\theta$, that point just becomes the vertical reflection of the corresponding point for $+\theta$. Meaning that its x-coordinate doesn't change. Meaning that its cosine doesn't change. And thus we have the following identity:

$$\cos(-\theta) = \cos(\theta)$$

By contrast, the y-coordinate of that point *does* change. It becomes negative of whatever it was before. And we know that the y-coordinate of that point is (by definition) $\sin(\theta)$. So we must have:

$$\sin(-\theta) = -\sin(\theta)$$

Here's another graph that shows it a bit better



Do you notice anything familiar about these two equations? They look just like the definition of even and odd functions! So cosine is an even function/a function that's horizontally symmetric around the y-axis (like x^2), and sine is an odd function (like x^3).

Tangent, meanwhile, will incorporate the negative.

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin(\theta)}{\cos(\theta)} = -\tan(\theta)$$

and thus will be odd. This is just like the question on that test—the product/quotient of an even function and an odd function is an odd function!

Interchange Identities

Another thing you've probably noticed, just from knowing the graphs of sine and cosine, is that they're the same—they've just got a slight horizontal shift. Why is this? The quick explanation (going back to the unit circle definition) is that circles have this nice radial symmetry, so that the x and y coordinates are changing at the same rate–the only difference is that the x-coordinate (cosine) starts at 1, and the y-coordinate (sine) starts at 0. Put differently, the x-coordinate is about $\pi/2$ radians ahead of where the y-coordinate is. Or:

$$\cos(\theta) = \sin(\theta + \pi/2)$$

Stated in the opposite way:

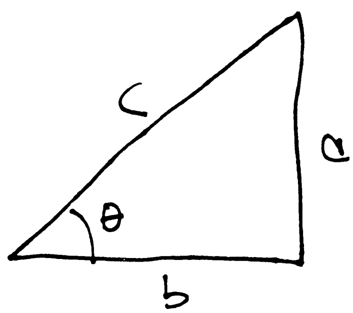
$$\sin(\theta) = \cos(\theta - \pi/2)$$

The Pythagorean Identity

There's another really cool relationship between trig functions. Namely: if I square the cosine of some angle, and then square the sine of the same angle, and add them, I just get one:

$$(\sin\theta)^2 + (\cos\theta)^2 = 1$$

Why is this true? Imagine we have a right triangle, with base lengths a and b, hypotenuse c, and an angle θ:



Then we know that

$$\sin(\theta) = \frac{a}{c}$$

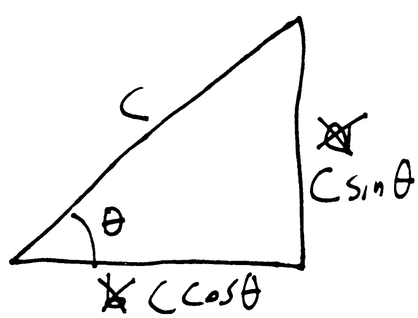
$$\cos(\theta) = \frac{b}{c}$$

Which, if we rearrange, is just another way of saying:

$$a = c\sin(\theta)$$

$$b = c\cos(\theta)$$

So we might as well label the sides of our triangle as $c\sin(\theta)$ and $c\cos(\theta)$, since they're just equal to the lengths:



But this is a right triangle, and so we must have:

$$\begin{aligned}
(c\sin\theta)^2 + (c\cos\theta)^2 &= c^2 \quad \text{(by the Pythagorean thm)} \\
c^2(\sin\theta)^2 + c^2(\cos\theta)^2 &= c^2 \quad \text{(distributing the square)} \\
(\sin\theta)^2 + (\cos\theta)^2 &= 1 \quad \text{(dividing by } c^2\text{)}
\end{aligned}$$

Often, by the way, we write things like $(\sin\theta)^2$ and $(\cos\theta)^2$ as $\sin^2\theta$ and $\cos^2\theta$, just as a more convenient notation (fewer parentheses!).

Note that we could write slightly modified versions of this identity. For example, if we divide both sides by $\cos^2(\theta)$, we get:

$$\begin{aligned}
\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta} &= \frac{1}{\cos^2\theta} \\
\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} &= \frac{1}{\cos^2\theta} \\
\tan^2\theta + 1 &= 1/\cos^2\theta
\end{aligned}$$

Or if we divide it all by $\sin^2\theta$:

$$\begin{aligned}
\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta} &= \frac{1}{\sin^2\theta} \\
\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} &= \frac{1}{\sin^2\theta} \\
1 + 1/\tan^2\theta &= 1/\sin^2\theta
\end{aligned}$$

For your convenience, I've summarized below all of the identities we've discussed. But please, please, don't try to memorize them! That will not help you understand trigonometry better! If you understand the trig—really *understand* it—then all of these identities[1] should make sense. They should be natural and obvious and you shouldn't even really have to think about these equations in order to apply them. This list, then, should not be a list of things you need to know, but a codification of things you already know.

Periodicity Identities (for any integer k):

- $\sin(\theta + 2k\pi) = \sin(\theta)$
- $\cos(\theta + 2k\pi) = \cos(\theta)$
- $\tan(\theta + k\pi) = \tan(\theta)$

Interchange Identities:

- $\cos(\theta) = \sin(\theta + \pi/2)$
- $\sin(\theta) = \cos(\theta - \pi/2)$

Symmetry Identities:

- $\cos(-\theta) = \cos(\theta)$
- $\sin(-\theta) = -\sin(\theta)$
- $\tan(-\theta) = -\tan(\theta)$

Pythagorean Identity:

$\sin^2\theta + \cos^2\theta = 1$

[1] With the possible exception of the Pythagorean identity, which is not obvious and takes a little work to derive.