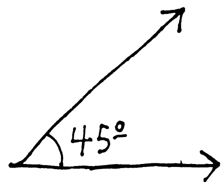
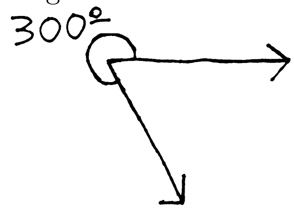
Radical Angle Measurement

Pre/Calculus 11, Veritas Prep.

How do we measure angles? We tend to measure them in degrees:

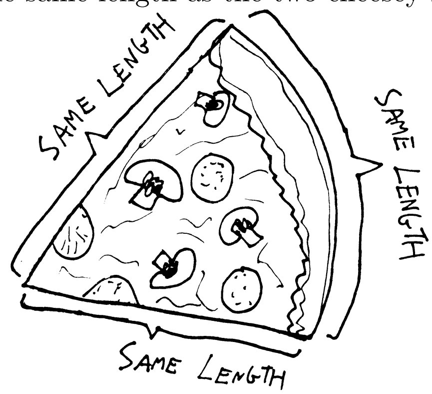




But why degrees? In a complete revolution of a circle, we have 360 degrees. Why 360? Isn't that kind of an awkward number? Why not define a "degree" such that all the degrees in a circle add up to 100 degrees? or 10 degrees?

Or maybe we could measure angles as a percent. We could say that the measure of the angle of a full circle is 100% (or 1), and then a 90° angle would be the same as a 25% (0.25) angle, a 135° angle would be equivalent to a 37.5% (0.375) angle, and so forth. This would be a natural way of measuring an angle—rather than being based on the arbitrariness of the number "360", it would use the much more natural choice of the number 1.

Another way to measure an angle—the way we'll use—is this. Today is Monday, and so it's pizza day at Veritas. In fact, it's the first Monday of the month, and so it's not only pizza day for students; it's free pizza day for faculty, too. And so imagine you're me, and you go down to the faculty office after this class to get your free pizza. And you have a strong predisposition for geometric symmetry, and so you want the crust of your piece of pizza to be the same length as the two cheesey sides, like so:

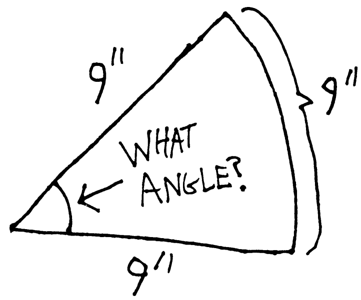


(Note that you measure the crust length as its outer length.) So you want, I guess, an equilateral piece of pizza. Sort of like an equilateral triangle, I suppose, except not a triangle *per se*, because the crust-side is curvy. But the same idea. So the question is: what angle should you cut this slice of pizza at, such that its crust length is the same as the length of the other two sides? (By angle I mean the angle at the center of the pizza, the one whose opposite side is the crust.)

Let's think about this. Imagine this is a pizza with an 18-inch diameter. Then it has a radius of 9 inches, and then we know the total length of its crust (measured on the outside) is $2\pi \cdot 9 = 18\pi$. We also know—and we'll slip back into degrees for a moment here, but I assure you it's in the service of a greater good—that the total number of degrees in this pizza is 360°. So we can think of this like a proportion:

$$\frac{\text{total crust length}}{\text{total degrees in pizza}} = \frac{\text{crust length of our slice}}{\text{degrees in our slice}}$$

Now, we know that, in our beautifully-equilateral slice, we want the length of the crust to be equal to the length of the other two sides. But because the pizza slice is just a slice of a circle, the other two sides must be both 9 inches long:



So if we plug in all that stuff, our proportion looks like this:

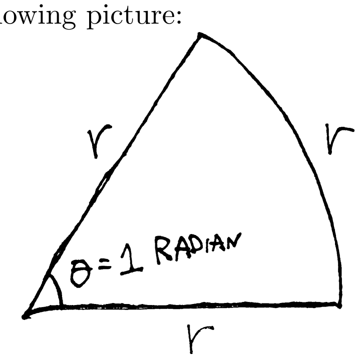
$$\frac{18\pi \text{ inches}}{360°} = \frac{9 \text{ inches}}{\text{degrees in our slice}}$$

Then it is easy to solve this and find out how many degrees there are in our equilaterial slice of pizza:

$$\begin{aligned}\text{degrees in our slice} \cdot \frac{18\pi}{360°} &= 9 \text{ inches}\\ \text{degrees in our slice} &= \frac{9 \text{ inches} \cdot 360°}{18\pi \text{ inches}}\\ \text{degrees in our slice} &\approx 57.9°\end{aligned}$$

So then if we want a beautiful, equilaterial slice of pizza, in which all the sides are the same length, we should cut it at an angle of 57.9°!!![1] Note something interesting here—that even though we started with a pizza of radius 9 inches, that all got cancelled out in our calculations. For *any* size pizza, an angle of 57.9° will give us an equilateral slice[2].

There is a point to all of this. Namely: this is how we will define the system of angle-measurement that we will use. We will create a measurement system in which the base unit is not 360° in a full revolution, or 100% in a full revolution, but rather, in which the base unit of measurement represents the angle needed to make the opposite arc of a circle equal to the length of its radii. Informally: we will define a **radian** as **the angle needed to make all three sides of a slice of pizza the same length**. Somewhat more formally, I could define it with the following picture:



By the way, note that the fact that a radian is approximately 57.9° should make sense. We know that an equilateral triangle has angles of 60°, and an equilateral pizza slice isn't quite the same thing—because of the curvy sides—but it is pretty close. So it makes sense that its angles should be close to 60°.

If we want to convert from degrees to radians (you would, of course, never want to convert out of radians!), we can simply derive a unit conversion factor. We know that the total number of radians in a full circle must be 2π, since that's how many times we can fit the radius of a circle around its circumference. (We would have a pizza whose crust-length was 2π times that of its slice-radius.) Likewise, we know that the total number of degrees in a full circle is 360. So we must have 360° for every 2π radians.

[1] Or rather, we should tell Mr. Fink to cut it for us with that angle.

[2] It wouldn't be too hard to prove this—just repeat all of the calculations we did, but using a pizza of radius r instead of radius 9

So, for example, if we want to convert 75° to radians:

$$\left(\frac{75 \text{ degrees}}{1}\right)\left(\frac{2\pi \text{ radians}}{360 \text{ degrees}}\right) = \frac{150\pi}{360} \text{ radians} = \frac{5\pi}{6} \text{ radians}$$

Problems

Rewrite the following angles in radians (or degrees, as appropriate):

1. 0

2. π

3. 2π

4. 3π

5. 4π

6. $\pi/6$

7. $5\pi/6$

8. $7\pi/6$

9. $11\pi/6$

10. $13\pi/6$

11. $\pi/4$

12. $3\pi/4$

13. $7\pi/4$

14. $9\pi/4$

15. $\pi/3$

16. $2\pi/3$

17. $4\pi/3$

18. $5\pi/3$

19. $7\pi/3$

20. $\pi/2$

21. $3\pi/2$

22. $5\pi/2$

23. $74,452\pi$

24. $8,000,000\pi$

25. $k\pi$

26. $a\pi/b$

27. $45°$

28. $60°$

29. $30°$

30. $90°$

31. $135°$

32. $110°$

33. $150°$

34. $170°$

35. $179°$

36. $225°$

37. $12°$

38. $5.34°$

39. $7°$

40. $180°$

41. $360°$

42. $365°$

43. (5 million)°

44. (your favorite number)°