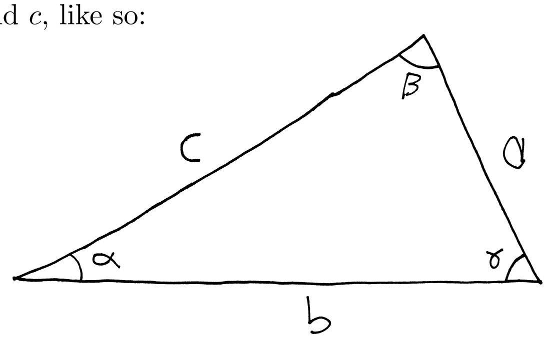
The Law of Cosines

Pre/Calculus 11, Veritas Prep.

You know the Pythagorean Theorem. $a^2+b^2=c^2$. It's pretty awesome. It turns *geometry* into *algebra*. But it only works for right triangles. You know what would be even more awesome? *A Pythagorean Theorem that worked for all triangles*!!!

So let's create one. Let's come up with a super-Pythagorean Theorem that relates the lengths of the sides of any triangle, even one that doesn't have a right angle. The usual name given to the theorem we'll come up with is the Law of Cosines, but that doesn't really do it justice. Call it the Super Pythagorean Theorem, or something. To put it more mathematically, the Law of Cosines is a *generalization* of the Pythagorean theorem, meaning that it does everything the Pythagorean Theorem does (relate the side lengths of a right triangle), but it does even more (relate the sides of *any* triangle). Likewise, the unit circle definition of trig functions is a generalization of the right-triangle definition of trig functions (because it does all the same stuff that the right-triangle definition does, but it also accounts for angles greater than 90° or less than 0°), and the rational numbers are a generalization of the integers (because the rationals include the integers, but also include other numbers, like 2/5 or 0.945). Or to use a non-math example, a cordless drill is a generalization of a screwdriver (because, given the right drill bits, you can use it as a screwdriver, but you can also use it to drill holes in masonry to pack dynamite into).

The theorem is this: imagine we have some triangle with angles α, β, and γ, and opposite sides (respectively) of length a, b, and c, like so:



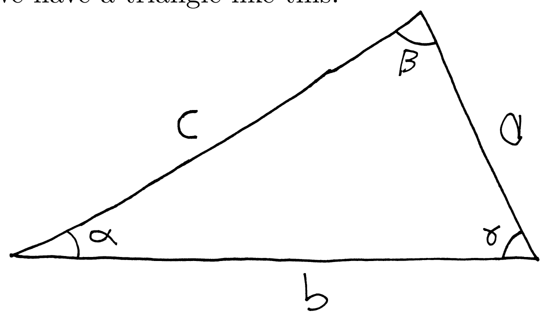
Then this equation is true:

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma)$$

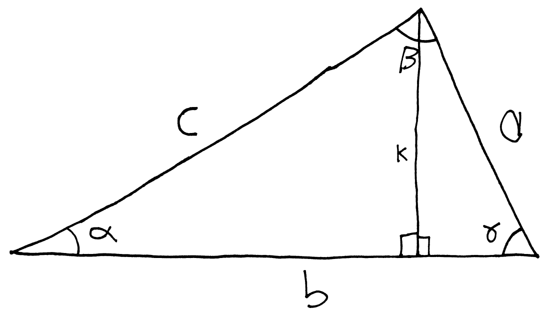
Proof: What is really cool about this proof is that it uses the Pythagorean Theorem. Meaning, it uses the Pythagorean Theorem to create a stronger version of itself. The strategy is to take a triangle (any triangle), split it up into right triangles, and apply the Pyth. Thm. to them. This is cool because, I mean, it's like, using simple tools to create complicated tools. So my historicizing comparison would be: civilization started with people banging stones together to create simple tools, and then using those tools to create slightly more complicated tools, etc. etc., and then ten thousand years later we have lasers and 747s. Which can all be traced back to people in Anatolia banging stones together.

Now, in daily life, I don't know how useful it is. I had never actually run across it until my fourth year at the U of C, when I learned (the day before) that I was supposed to be teaching it. But I got so excited, because it is SO COOL. Because it's a generalization of this awesome theorem that we use all the time. If there are any fundamental themes or ideas behind "mathematics," this idea of *generalization* is one of them. It is this idea of taking what we know and trying to think about it more broadly. What if we *can* take the square root of negative numbers? What happens? The number system that results (the complex numbers) not only contains everything we already know about the real numbers, but has so much more, and has so much additional, beautiful structure, that we'd never see if we refused to look for it and simply plugged our ears and clenched our eyes shut and shouted, "Of course you can't take the square root of a negative number!"

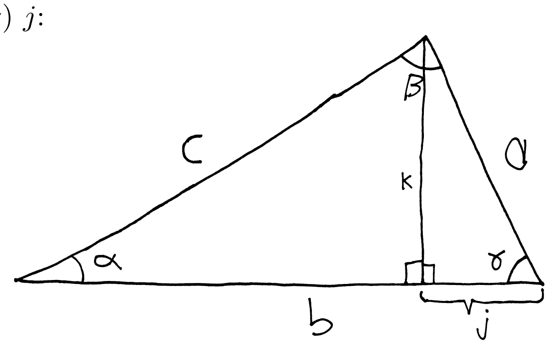
Right. So, the proof. Imagine we have a triangle like this:



And we split it up into two right triangles by drawing a line from angle β, and let's say that line has length k:



Then: consider that right triangle on the right. Let's call the length of that bottom side of that triangle (the side adjacent to γ) j:



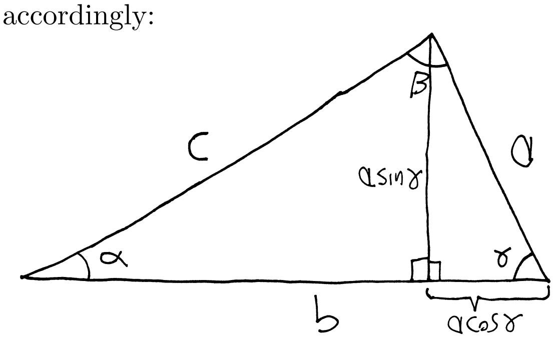
Now let's see if we can find the lengths of k and j, only in terms of what we already know. We know that

$$\sin(\gamma) = \frac{k}{a} \qquad \text{and} \qquad \cos(\gamma) = \frac{j}{a}$$

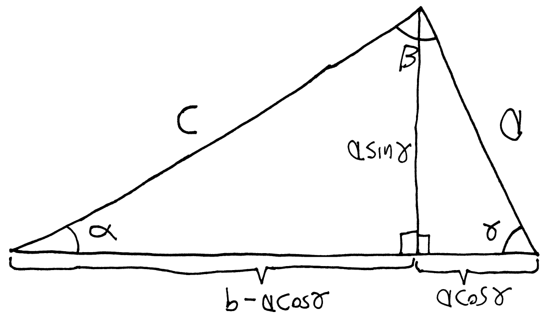
which are just different ways of saying that

$$k = a\sin(\gamma) \qquad \text{and} \qquad j = a\cos(\gamma)$$

So we can relabel our triangle accordingly:



But now let's consider the right triangle on the left. We know that it has a hypotenuse of c, and that one of its sides (the side opposite angle α) is of length $a\sin(\gamma)$. But what about the side on the bottom? What is its length? We know that since the total length of the bottom line (for both triangles) is b, and that the length of the right-hand portion is $a\cos(\gamma)$, the length of the portion on the left right triangle must be $b - a\cos(\gamma)$:



But now we can apply the Pythagorean Theorem to that right triangle on the left!!! We must have:

$$
\begin{aligned}
c^2 &= (b - a\cos\gamma)^2 + (a\sin\gamma)^2 \\
&= (b^2 - 2ab\cos\gamma + a^2\cos^2(\gamma)\,) + a^2\sin^2(\gamma) && \text{(squaring)} \\
&= b^2 - 2ab\cos\gamma + a^2(\cos^2\gamma + \sin^2\gamma) && \text{(factoring last couple terms)} \\
&= b^2 - 2ab\cos\gamma + a^2(1) && \text{(Pythagorean identity!)} \\
&= a^2 + b^2 - 2ab\cos(\gamma) && \text{(rearranging)}
\end{aligned}
$$

Yay! Note that, if $\gamma = 90^\circ = \pi/2$, this just reduces to the good old Pythagorean Theorem:

$$c^2 = a^2 + b^2 - 2ab\cos(\pi/2) = a^2 + b^2 - 2ab \cdot 0 = a^2 + b^2$$

So don't memorize the Pythagorean Theorem—memorize THIS!!!