Transfinitude

Math 12, Veritas Prep.

> Now think of the following use of language: I send someone shopping. I give him a slip marked "five red apples." He takes the slip to the shopkeeper, who opens the drawer marked "apples"; then he looks up the word "red" in a table and finds a color sample opposite it; then he says the series of cardinal numbers—I assume that he knows them by heart—up to the word "five" and for each number he takes an apple of the same colour as the sample out of the drawer.—It is in this and similar ways that one operates with words.—-"But how does he know where and how he is to look up the word 'red' and what is he to do with the word 'five'?"—Well, I assume that he *acts* as I have described. Explanations come to an end somewhere.—But what is the meaning of the word "five"?—No such thing was in question here, only how the word "five" is used. — Ludwig Wittgenstein, *Philosophical Investigations*, I.1

> The Cantorian infinite... is of the greatest and most fundamental importance; the understanding of it opens the way to whole new realms of mathematics and philosophy.
> —Bertrand Russell, *Introduction to Mathematical Philosophy*, p.65

If we have a set, one of the most basic questions we can ask about it is: how many elements does it have? If we are dealing with finite sets, the answer to this question is obvious and boring. We simply count the number of items in the set. But what's counting? To count, I suppose, is to match up every element in a set with one of the natural numbers, in order. I point to each element in the set in turn, and each time I point I say the next natural number. Right? This is not something new. We've been counting all of our lives; I'm just trying to describe that process. The fancy math-word is **cardinality**, as in "the cardinality of the set $\{a, b, c\}$ is 3." We usually denote cardinality with square bars around the set (think absolute value, another measure of magnitude!), e.g., if $S = \{a, b, c\}$, then $|S| = 3$.

So if I have the set of all math teachers at Veritas, and I want to know the cardinality of that set (i.e., how many math teachers there are), I simply match up each math teacher with a natural number, and the largest natural number I reach is the number of math teachers at Veritas:

$1 \leftrightarrow$ Alexander
$2 \leftrightarrow$ Austin
$3 \leftrightarrow$ Baldwin
$4 \leftrightarrow$ Mapes
$5 \leftrightarrow$ Scharber

The natural numbers continue past 5, but I'm out of math teachers. So the number of math teachers at Veritas is 5. Or, put more formally, the cardinality of the set of math teachers at Veritas is 5.

But here's the question: what if I have a set that contains not a finite number of elements, but an infinite number? how can I find its cardinality? does that question even make any sense? does it make any sense to talk about infinite quantities? aren't they all the same size (infinite)?

Well, by our definition above, we said that a set has cardinality k if we can put it in a one-to-one correspondence with the first k natural numbers. So we measure cardinality by *comparison*—by comparing the number of elements a set has with some subset of the natural numbers. Even more generally, we can say that two sets have the same cardinality if and only if they can be put into a one-to-one correspondence with *each other* (i.e., if every element of set A can be paired up with exactly one element of set B). (Fancy math word for that: a **bijection**.) So, for example, I could say that there are the same number of math teachers at Veritas as there are major metropolitan areas in New York State, and I know that because I can make a one-to-one correspondence between the two sets:

$$\begin{aligned}
\text{New York City} &\leftrightarrow \text{Alexander}\\
\text{Albany} &\leftrightarrow \text{Austin}\\
\text{Syracuse} &\leftrightarrow \text{Baldwin}\\
\text{Rochester} &\leftrightarrow \text{Mapes}\\
\text{Buffalo} &\leftrightarrow \text{Scharber}
\end{aligned}$$

But what if we are dealing with infinite sets? Take, for example, the natural numbers and the integers. They're both infinite. But are they the same size? Do they have the same number of elements? $\mathbb{Z}$ includes the negatives, which aren't in $\mathbb{N}$, and so does that mean that $\mathbb{Z}$ has more elements than $\mathbb{N}$? Or do they both have the same number of elements, because they're both infinite?

Based on our definition of cardinality, $|\mathbb{Z}| = |\mathbb{N}|$. That is to say: *there are the same number of integers as there are natural numbers.* Why? Because we can put them in a one-to-one correspondence, like so:

$$\begin{array}{rcccccccc}
\mathbb{N}: & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\
 & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
\mathbb{Z}: & 0 & 1 & -1 & 2 & -2 & 3 & \cdots
\end{array}$$

In fact, I could even say that there are the same number of natural numbers as there are natural numbers greater than 5, because, again, I can put them in a one-to-one correspondence:

$$\begin{array}{rcccccccc}
\mathbb{N}: & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\
 & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
\mathbb{N}^{>5}: & 6 & 7 & 8 & 9 & 10 & 11 & \cdots
\end{array}$$

Here's another example:

> A group of mathematicians are road-tripping to a set theory conference, and they need to find somewhere to stay that night. They pass a billboard on the highway: The Hilbert Hotel! An Infinite Number of Rooms!!! So they figure: well, it has an infinite number of rooms—surely it must have a vacancy! So they stop, but the clerk at the hotel says: "I'm very sorry, but we're all out of rooms—an infinite number of tour buses just pulled up, with an infinite number of guests. You'll have to find somewhere else to stay.
>
> "That won't be necessary," says one of the mathematicians. "Instead, why don't you ask the guest staying in Room 1 to move to Room 2, the guest staying in Room 2 to move to Room 3, the guest staying in Room 3 to move to Room 4, and so on. That way, everyone will still have a room—since you have an infinite number of them—but it'll free up Room 1. We'll take that one."

So perhaps this answers our question—every infinite set can be put in a one-to-one correspondence with $\mathbb{N}$; therefore, every infinite set has the same cardinality, i.e., infinity has but one "size."

But what if we consider another infinite set—namely, the set of the real numbers, $\mathbb{R}$. (Recall that the real numbers are those numbers that we can express as an infinitely-long decimal, either with or without a pattern to the digits—for example, $\pi = 3.14159\ldots$ or $3/2 = 1.50000\ldots$.) Can we put $\mathbb{R}$ into a one-to-one correspondence with $\mathbb{N}$? As it turns out, we can't. Put differently: *there are different sizes of infinity.* "The essence of Cantor's result is that there are (at least) two distinct *types* of infinity: one kind of infinity describes how many entries there can be in an infinite directory or table, and another describes how many real numbers there are (i.e., how many points there are on a line, or a line segment)—and this latter is "bigger", in the sense that the real numbers cannot be squeezed into a table whose length is described by the former kind of infinity."[1]

[1] *Gödel, Escher, Bach: an Eternal Golden Braid, by Douglas Hofstadter (1979)*, p. 421

Imagine, for the sake of argument, that we *could* enumerate the real numbers. That is to say, imagine that we could write them in a list, such that we could match up each real number with a corresponding natural number. And, just to make things simpler, let's say that we only want to match up the real numbers between 0 and 1 with the naturals (because if we could do that, we could, using a similar method, match up all the reals). Exactly how we would do this is unclear, but let's imagine the beginning of our list of all the real numbers looks like this:

$$\begin{aligned}
1 &\leftrightarrow .\mathbf{9}34579\ldots \\
2 &\leftrightarrow .3\mathbf{6}9567\ldots \\
3 &\leftrightarrow .55\mathbf{4}531\ldots \\
4 &\leftrightarrow .108\mathbf{7}76\ldots \\
5 &\leftrightarrow .4191\mathbf{1}2\ldots \\
\vdots & \qquad \vdots
\end{aligned}$$

What Cantor proved was that no matter how we create such a list, *there will always be a real number missing from it.* Why? Well, notice that "the digits that run down the diagonal are in boldface:

$$9, 6, 4, 7, 1 \ldots$$

"Now those diagonal digits are going to be used in making a special real number d which is between 0 and 1 but which, we will see, is not in the list. To make d, you take the diagonal digits in order, and change each one of them by some other digit. When you prefix this sequence of digits by a decimal point, you have d. There are of course many ways of changing a digit to some other digit, and correspondingly many different d's. Suppose, for example, that we *subtract* 1 *from the diagonal digits* (with the convention that 1 taken from 0 is 9). Then our number d will be:

$$d = .85360\ldots$$

"Now, because of the way we constructed it:

- d's first digit is not the same as the first digit of the first number on the list;
- d's second digit is not the same as the second digit of the second number on the list;
- d's third digit is not the same as the third digit of the third number on the list;
- . . . and so on.

"Hence,

- d is different from the first number on the list;
- d is different from the second number on the list;
- d is different from the third number on the list;
- . . . and so on

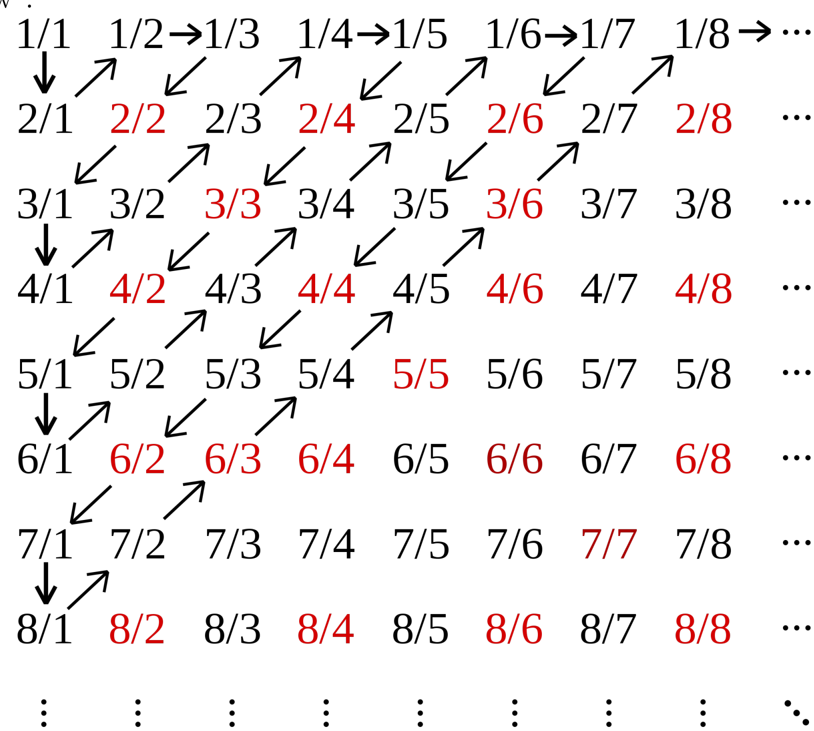
"In other words, d is not on the list! [Put differently: there are more real numbers than there are natural numbers!!!]

"... At first, the Cantor argument might not seem fully convincing. Isn't there some way to get around it? Perhaps by throwing in the diagonally constructed number d, one might obtain an exhaustive list. If you consider this idea, you will see that it helps not a bit to throw in the number d, for as soon as you assign it to a specific place in the table, the diagonal method becomes applicable to the new table, and a new missing number d' can be constructed, which is not yet in the new table. No matter how many times you repeat the operation of constructing a number by the diagonal method and then throwing it in to make a "more complete" table, you still are caught on the ineradicable hook of Cantor's method. You might even try to build a table of reals which tries to outwit the Cantor diagonal method by taking the whole trick, lock, stock, and barrel, including its insidious repeatability, into account somehow. It is an interesting exercise. But if you tackle it, you will see that no matter how you twist and turn trying to avoid the Cantor 'hook," you are still caught in it. One might say that any self-proclaimed "table of all reals" is hoist by its own petard."[2]

Quick note on notation and terminology before we get much further: we say that any set with the same cardinality as the natural numbers has cardinality $\aleph_0$ (read: "aleph-not"), with $\aleph$ being the first letter of the Hebrew alphabet[3]. Sets that we can put in this one-to-one correspondence with $\mathbb{N}$ we refer to as **denumerable** or **enumerable** or **countable** (so a set that we can put in a correspondence with $\mathbb{N}$ is either **finite** or **countably infinite**.) Sets whose cardinality is greater than $\aleph_0$ we call **uncountable**. The reals, for instance, are uncountable, and to denote their cardinality we usually use a lowercase c: $|\mathbb{R}| = c$.

So if you want to be one of those obnoxious people who corrects people by saying things like "Well, *technically*, you didn't call me on my birthday, since you called me at 12:45 AM the next day," you can now add to your arsenal phrases like: "Well, *technically*, we haven't had this same argument 'uncountably many' times, since it would be possible to place the the times we've had this argument into a bijection with the natural numbers (a finite bijection, no less)."

Now, maybe your response to the fact that $|\mathbb{R}| > |\mathbb{N}|$ is, "So what? Of course there are more real numbers than natural numbers. After all, there are an infinite number of reals between, say, three and five, but only one natural (four). They're a lot denser on the number line." So let me give you another example: what about $\mathbb{Q}$? Are there more rational numbers than natural numbers? or are there the same number? Like with the reals, there are an infinite number of rational numbers between any two natural numbers. There are an infinite number of rationals between 3 and 5, but only one natural number (4). Yet—as it turns out—there are the same number of rationals as there are naturals—we *can* enumerate the rationals. Here's how[4]:



[2] *Gödel, Escher, Bach: an Eternal Golden Braid*, by Douglas Hofstadter (1979), pp.421-424

[3] Cantor thought that these ideas were so important that they needed not just a new Roman or Greek letter—but a letter from an entirely new alphabet. He was right.

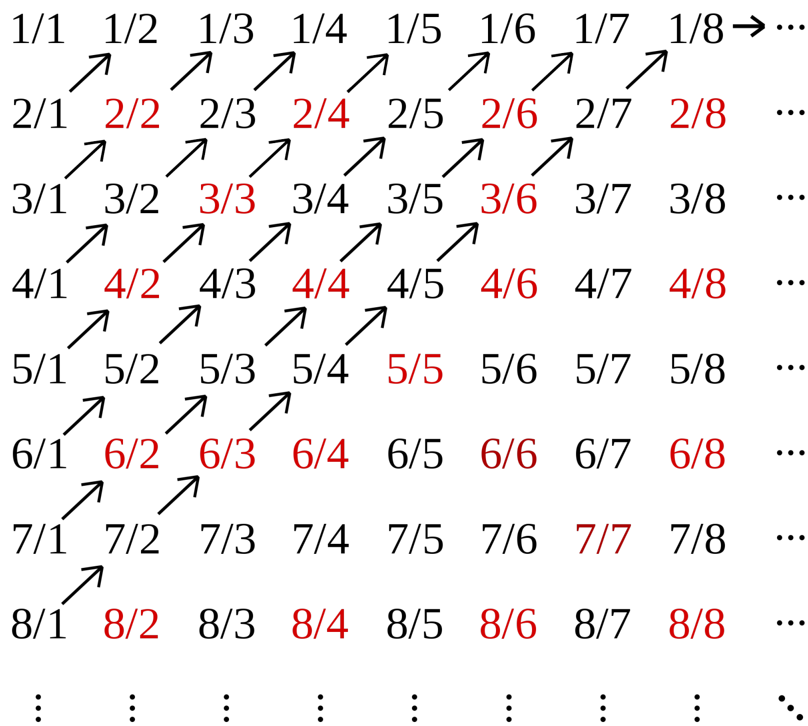
[4] `http://en.wikipedia.org/wiki/File:Diagonal_argument.svg`

Take a careful look at this diagram and try to figure out what's going on. You should convince yourself that 1) every rational number indeed exists somewhere on this grid, and 2) our snake-like list will eventually hit every one.

Here's another way to phrase it (pay attention to the sequence of numbers in the numerator and in the denominator):

$$\begin{array}{cccccccccccccccc} \mathbb{N}: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & \cdots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ \mathbb{Q}: & \frac{1}{1} & \frac{2}{1} & \frac{1}{2} & \frac{1}{3} & \frac{2}{2} & \frac{3}{1} & \frac{4}{1} & \frac{3}{2} & \frac{2}{3} & \frac{1}{4} & \frac{1}{5} & \frac{2}{4} & \frac{3}{3} & \frac{4}{2} & \cdots \end{array}$$

Actually, here—what if I just rearrange the arrows, so that rather than snaking across the grid like a snake, the list of rational numbers always moves towards the upper-right? Whenever you reach the top, you go back down to the lower left:



And so if we actually write out the correspondence, it looks like this:

$$\begin{array}{cccccccccccccccc} \mathbb{N}: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & \cdots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ \mathbb{Q}: & \frac{1}{1} & \frac{2}{1} & \frac{1}{2} & \frac{3}{1} & \frac{2}{2} & \frac{1}{3} & \frac{4}{1} & \frac{3}{2} & \frac{2}{3} & \frac{1}{4} & \frac{5}{1} & \frac{4}{2} & \frac{3}{3} & \frac{2}{4} & \cdots \end{array}$$

Maybe if I strategically throw in some commas, it will make sense (with the presumptive next number in brackets):

$$\text{numerators} : 1, 2\,1, 3\,2\,1, 4\,3\,2\,1, 5\,4\,3\,2\,[1]\ldots$$
$$\text{denominators} : 1, 1\,2, 1\,2\,3, 1\,2\,3\,4, 1\,2\,3\,4\,[5]\ldots$$

So there are the same number of rational numbers and integers as there are natural numbers—but there are more real numbers!

Let's take this even further. We think of numbers, at least the natural numbers, as representing a sort of *magnitude*—or, differently put, as representing some quantity that we can combine with other quantities. We can add three to five, and get eight. But what if I consider these transfinite cardinalities to be numbers themselves? If I can count collections of twelve objects using the first 12 natural numbers, why can't I count collections of $\aleph_0$ objects using all of $\mathbb{N}$? And why can't I then consider $\aleph_0$ to be a number in its own right? Well, I can. We refer to numbers constructed in this way as **transfinite cardinals** (or **transfinite cardinal numbers**) (as opposed to **finite cardinals**, which are just the natural numbers).

I might ask: what's $\aleph_0 + 1$? This, I suppose, will just be the cardinality of a set that has one more element than the natural numbers. (Formally, I can think of $\aleph_0 + 1 = |\mathbb{N} \cup \{\mathbb{N}\}|$, in the same way that we defined arithmetic for finite natural numbers—to add one, I just adjoin (fancy name for "union") a set containing one element—in this case, the set of the set of natural numbers, which contains just one element... the set of natural numbers.) But that will just have a cardinality $\aleph_0$: for example,

$$\begin{array}{rcccccccc} \mathbb{N}: & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ \mathbb{N} \cup \{\mathbb{N}\}: & \{\mathbb{N}\} & 1 & 2 & 3 & 5 & 5 & \cdots \end{array}$$

So then we must have $\aleph_0 + 1 = \aleph_0$. As Russell puts it, "The most noteworthy and astonishing difference between a [finite] number and this new [transfinite] number is that this new number is unchanged by adding 1 or subtracting 1 or doubling or halving or any of a number of other operations which we think of as necessarily making a number larger or smaller."[5]

What if I want to find $\aleph_0 + \aleph_0$? That is, what if I want to know the cardinality of the union of two sets, each with cardinality $\aleph_0$? Suggestion: imagine you have sets A and B, with $|A| = |B| = \aleph_0$, and

$$A = \{a_1, a_2, a_3, \cdots\}$$

$$B = \{b_1, b_2, b_3, \cdots\}$$

Can you put $A \cup B$ into a one-to-one correspondence with the natural numbers? What about $\aleph_0 + \aleph_0 + \aleph_0$? What about $k \cdot \aleph_0$—that is, what's the cardinality of the union of k sets, each of cardinality $\aleph_0$?

What if you have countably many countable sets? That is to say, what if you have $\aleph_0$ sets, each with cardinality $\aleph_0$? What's the cardinality of the union of all of them? That is to say, what is $\aleph_0 \cdot \aleph_0$? Suggestion: consider the sets

$$\begin{aligned} S_1 &= \{s_{1,1}, s_{1,2}, s_{1,3}, \cdots\} \\ S_2 &= \{s_{2,1}, s_{2,2}, s_{2,3}, \cdots\} \\ S_3 &= \{s_{3,1}, s_{3,2}, s_{3,3}, \cdots\} \\ S_4 &= \{s_{4,1}, s_{4,2}, s_{4,3}, \cdots\} \\ \vdots \quad & \quad \vdots \end{aligned}$$

Can you enumerate the elements of these sets? (Hint: look at the proof of $|\mathbb{Q}| = \aleph_0$.)

So it seems like no matter what I do to $\aleph_0$, I always end up with just $\aleph_0$. Which, while different than finite arithmetic, seems rather boring. And yet, as our proof that $|\mathbb{R}| \neq |\mathbb{N}|$ shows, there must be other infinities other than $\aleph_0$! There must be greater infinities! So how do we get to them? USING A POWER SET! If I take the power set of a set with cardinality $\aleph_0$, the resulting set does **not** also have cardinality $\aleph_0$. Here's why:

[5] Bertrand Russell, *Introduction to Mathematical Philosophy* (1993 Dover reprint of 1920 original), p. 79

Theorem (Cantor's Theorem): For any set X, the cardinality of $P(X)$ is greater than the cardinality of X.
Proof: To get a handle on the proof, let's examine it for the specific case when X is countably infinite. Without loss of generality, we may take $X = \mathbb{N} = \{1, 2, 3, \dots\}$, the set of natural numbers.

Suppose that $\mathbb{N}$ is bijective with its power set $P(\mathbb{N})$. Let us see a sample of what $P(\mathbb{N})$ looks like:

$$P(\mathbb{N}) = \{\varnothing, \{1,2\}, \{1,2,3\}, \{4\}, \{1,5\}, \{3,4,6\}, \{2,4,6,\dots\}, \dots\}$$

$P(\mathbb{N})$ contains infinite subsets of $\mathbb{N}$, e.g. the set of all even numbers $\{2, 4, 6, ...\}$, as well as the empty set.

Now that we have a handle on what the elements of $P(\mathbb{N})$ look like, let us attempt to pair off each element of $\mathbb{N}$ with each element of $P(\mathbb{N})$ to show that these infinite sets are bijective. In other words, we will attempt to pair off each element of $\mathbb{N}$ with an element from the infinite set $P(\mathbb{N})$, so that no element from either infinite set remains unpaired. Such an attempt to pair elements might look like this:

$$\mathbb{N}\left\{\begin{matrix} 1 & \longleftrightarrow & \{4,5\} \\ 2 & \longleftrightarrow & \{1,2,3\} \\ 3 & \longleftrightarrow & \{4,5,6\} \\ 4 & \longleftrightarrow & \{1,3,5\} \\ \vdots & \vdots & \vdots \end{matrix}\right\}P(\mathbb{N})$$

Given such a pairing, some natural numbers are paired with subsets that contain the very same number. For instance, in our example the number 2 is paired with the subset $\{1, 2, 3\}$, which contains 2 as a member. Let us call such numbers **selfish**. Other natural numbers are paired with subsets that do not contain them. For instance, in our example the number 1 is paired with the subset $\{4, 5\}$, which does not contain the number 1. Call these numbers **non-selfish**. Likewise, 3 and 4 are non-selfish.

Using this idea, let us build a special set of natural numbers. This set will provide the proof by contradiction we seek.

Let D = the set of all non-selfish natural numbers

By definition, the power set $P(\mathbb{N})$ contains all sets of natural numbers, and so it contains this set D as an element. Therefore, in our bijection above between $\mathbb{N}$ and $P(\mathbb{N})$, D must be paired off with some natural number. Let's call that number d. However, this causes a problem. If d is in D, then d is selfish because it is in the corresponding set. If d is selfish, then d cannot be a member of D, since D was defined to contain only non-selfish numbers. But if d is not a member of D, then d is non-selfish and must be contained in D, again by the definition of D. (Read this paragraph several times until you understand it.)

This is a contradiction because the natural number d must be either in D or not in D, but neither of the two cases is possible. Therefore, there is no natural number which can be paired with D, and we have contradicted our original supposition, that there is a bijection between $\mathbb{N}$ and $P(\mathbb{N})$.

Note that the set D may be empty. This would mean that every natural number x maps to a set of natural numbers that contains x. Then, every number maps to a nonempty set and no number maps to the empty set. But the empty set is a member of $P(\mathbb{N})$, so the mapping still does not cover $P(\mathbb{N})$.

Through this proof by contradiction we have proven that the cardinality of $\mathbb{N}$ and $P(\mathbb{N})$ cannot be equal. We also know that the cardinality of $P(\mathbb{N})$ cannot be less than the cardinality of ℕ because $P(\mathbb{N})$ contains all singletons, by definition, and these singletons form a "copy" of ℕ inside of $P(\mathbb{N})$. Therefore, only one possibility remains, and that is the cardinality of $P(\mathbb{N})$ is strictly greater than the cardinality of ℕ, proving Cantor's theorem.[6]

One of the consequences of the fact that $|P(X)| > |X|$ is that, simply by successively taking power sets, we can make bigger and bigger sets *ad infinitum.* And we can even make bigger and bigger infinite sets!!! Put differently: *there are an infinite number of infinities.* Notationally, we usually write them by changing the subscript on the $\aleph$, i.e.,

$$\begin{aligned} |\mathbb{N}| &= \aleph_0 \\ |P(\mathbb{N})| &= \aleph_1 \\ |P(P(\mathbb{N}))| &= \aleph_2 \\ |P(P(P(\mathbb{N})))| &= \aleph_3 \\ \vdots \ &\ \vdots \end{aligned}$$

SO THERE ARE AT LEAST COUNTABLY-INFINITE TRANSFINITE CARDINALS!!! At least countably many infinities! THE WORLD IS SO WEIRD!!!! (Are there more transfinite cardinals than just these? Is there some $\aleph_k$ that can't be generated simply by power-setting ℕ?

We might ask: where does $|\mathbb{R}|$ fall? We've proven that $|\mathbb{R}| > |\mathbb{N}|$, but is $|\mathbb{R}| = \aleph_1$? or $\aleph_2$? or what? is this set of $\{\aleph_0, \aleph_1, \aleph_2, \aleph_3, \dots\}$ the only transfinite numbers there are? or are there other transfinites that we haven't included?

From the Stanford Encyclopedia of Philosophy[7]:

> The smallest infinite cardinal is the cardinality of a countable set. The set of all integers is countable, and so is the set of all rational numbers. On the other hand, the set of all real numbers is uncountable, and its cardinal is greater than the least infinite cardinal. A natural question arises: is this cardinal (the continuum) the very next cardinal? In other words, is it the case that there are no cardinals between the countable and the continuum? As Cantor was unable to find any set of real numbers whose cardinal lies strictly between the countable and the continuum, he conjectured that the continuum is the next cardinal: the **Continuum Hypothesis**. Cantor himself spent most of the rest of his life trying to prove the Continuum Hypothesis and many other mathematicians have tried too. One of these was David Hilbert, the leading mathematician of the last decades of the 19th century. At the World Congress of Mathematicians in Paris in 1900 Hilbert presented a list of major unsolved problems of the time, and the Continuum Hypothesis was the very first problem on Hilbert's list.[8]

Put differently, we have **The Continuum Hypothesis:** there is no set whose cardinality is between that of the naturals and that of the real numbers, i.e., there is no set S such that $\aleph_0 < |S| < \aleph_1$.

> ... Whether it's what unhinged him or not is an unanswerable question, but it is true that his inability to prove the C.H. caused Cantor pain for the rest of his life; he considered it his great failure. This too, in hindsight, is sad, because professional mathematicians now know exactly why G. Cantor could neither prove nor disprove the C.H. The reasons are deep and important

[6] "Cantor's theorem," *Wikipedia*, `http://en.wikipedia.org/w/index.php?title=Cantor's_theorem`. And before you start complaining about how I'm just hacking my notes from Wikipedia, let me say that this proof was, for math, very well-written, and Mr. Alexander spends so much time writing these damn notes as it is that he figured there was no point in doing redundant work—look, it's half past midnight! I have these notes, along with integration notes and "basic algebraic ideas" notes to finish! And I was going to write my cousin an email! I was going to go to a concert tonight!

[7] which, by the way, is a fantastic resource—it's an online encyclopedia of everything philosophical, but (unlike Wikipedia) written by actual scholars and edited by actual editors and fact-checkers.

[8] Jech, Thomas, "Set Theory", *The Stanford Encyclopedia of Philosophy*, `http://plato.stanford.edu//entries/set-theory/`

and go corrosively to the root of axiomatic set theory's formal Consistency, in rather the same way that K. Gödel's Incompleteness proof deracinate all math as a formal system. Once again, the issues here can be only sketched or synopsized...

The Continuum Hypothesis and the aforementioned Axiom of Choice are the two great besetting problems of early set theory. Particularly respecting the former, it's important to distinguish between two different questions. One, which is metaphysical, is whether the C.H. is true or false. The other is whether the C.H. can be formally proved or disproved from the axioms of standard set theory.

(These two questions collapse into one only if either (1) formal set theory is an actual map/mirror of the actual reality of ∞ and ∞-grade sets, or (2) formal set theory *is* that actual reality, meaning that a given infinite set's 'existence' is all and only a matter of its logical computability with the theory's axioms. Please notice that these are just the questions about the metaphysical status of abstract entities that have afflicted math since the Greeks.)

It's the second question that has been definitively answered, over a period of several decades, by K. Gödel and P. Cohen, to wit:

1938—Gödel formally proves that the general form of the Continuum Hypothesis is Consistent with the axioms of ZFC—i.e., that if the C.H. is treated as its own axiom and added to those of set theory no logical contradiction can possibly result.

1963—In one of those out-of-nowhere *coups d'etats* that pop scholars and moviemakers love, a young Stanford prof. named Paul J. Cohen proves that the *negation* of the general C.H. can be added to ZFC without contradiction.

These two results together establish what's now known as the *Independence of the Continuum Hypothesis*, meaning that the C.H. occupied a place rather like the Paralell Axiom's w/r/t Euclidean geometry: it can be neither proved nor disproved from set theory's standard axioms... This kind of Independence (which can also be called Undecidability) is a big deal indeed. For one thing, it demonstrates that Gödel's Incompleteness results (as well as A. Church's 1936 proof that 1st-order predicate logic is also Undecidable) are not just describing theoretical possibilities, that there really are true and significant theorems in math that can't be proved/disproved. Which in turn means that even a maximally abstract, general, wholly formal mathematics is not going to be able to represent (or, depending on your metaphysical convictions, contain) all real-world mathematical truths. It's this shattering of the belief that 100% abstraction = 100% truth that pure math has still not recovered from—nor is it yet even clear what 'recovery' here would mean.[9]

[9] *Everything and More: A Compact History of* ∞, by David Foster Wallace (W.W. Norton, 2003), pp300-2, with some footnotes absorbed into the body