Numbers

Math 12, Veritas Prep.

What's in a number? This is a hard question to answer—is it answerable?—and I certainly don't intend to do so here. However, I do wish to present at least one way in which we can formally define the natural numbers. I'm going to show a way in which we can create $\mathbb{N}$ out of just sets, and then be able to define and create and construct and learn about addition and multiplication and all of our beloved arithmetic—founded all on just sets.

The basic idea, as I'll demonstrate in a moment, is to start with the empty set, and then by makign new sets containing the empty set, be able to create different sets with as many elements as we like (and then define, e.g., "twelve" as the set created in this way that has 12 elements.)

So. We define zero to be the empty set—the set with no elements. And then, we would like to make the number 1 out of nothing but what we already have (zero), and so we define 1 as being the set containing zero, i.e., the set containing the empty set:

$$1 = \{0\}$$

Note, then, that 1 is not equal to the empty set. 1 is a set that contains one element. That element is another set, and that set happens to be the empty set. But 1 is itself not empty. If this bothers you, try writing the empty set just as $\emptyset = \{\}$. Then $0 = \{\}$, and

$$\begin{aligned} 0 &= \{\ \} \\ 1 &= \{0\} = \{\ \{\}\ \} \end{aligned}$$

Next, we define 2 as the set containing both zero and one:

$$2 = \{0, 1\} = \{\emptyset, \{\emptyset\}\ \}$$

Again, note that by this definition, "2" has exactly two elements. Continuing in this vein, we define 3 as the set containing 0, 1, and 2:

$$3 = \{0, 1, 2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\ \}$$

And we continue:

$$\begin{aligned} 0 &= 0 = \emptyset \\ 1 &= \{0\} = \{\emptyset\} \\ 2 &= \{0, 1\} = \{\emptyset, \{\emptyset\}\ \} \\ 3 &= \{0, 1, 2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\ \} \\ 4 &= \{0, 1, 2, 3\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\ , \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\ \}\} \\ 5 &= \{0, 1, 2, 3, 4\} = \text{etc.} \end{aligned}$$

"The idea is simply to define a natural number n as the set of all smaller natural numbers: $\{0, 1, \ldots, n-1\}$. In this way, n is a particular set of n elements."[1] And we do this by building a universe of *pure sets* in which we have sets of sets of sets, and the fundamental element, when we unpack enough of the sets, is *nothing*—just the empty set. We found our idea of numbers on nothing! It is very beautiful and very cool. The idea—why we base it on $\{\ \}$ and not on, say, {cheese}—because we could have gone through this same procedure, starting with any set, and gotten sets with a fixed number of elements—the reason we base it on the empty set because we want our idea of numbers to use as few primitive ideas as possible. We don't want our idea of numbers to be based on some ontologically-questionable set, like $\{x | x \text{ is a cheese}\}$.

Somewhat more formally, we can create the natural numbers in this way (This is the same thing we just did; I'm just codifying it here):

[1] "Basic Set Theory," *Stanford Encyclopedia of Philosophy*, `http://plato.stanford.edu/entries/set-theory/primer.html`

1. we define 0 as $\emptyset$, and
2. for any natural number n, we define $n+1$ as $n+1 = n \cup \{n\}$.

So we start with zero, and to create one, we adjoin (fancy name for "union") zero to the set containing itself—i.e., we simply add on another single element, and the most natural choice for that single element to add on is a copy of the set itself.

In principal, then, we can use this definition of a natural number to derive all of our basic properties of arithmetic. We just need to define what it means to add any two numbers, which we can do (again, recursively):

$$n + (m+1) = (n+m) \cup \{n+m\}$$

and likewise define multiplication, which we can do by setting up similar axioms:

$$n \cdot 0 = 0 \text{ and } n \cdot (m+1) = (n \cdot m) + n$$

Unfortunately the disadvantage with these is that they're both recursive definitions—that we're defining not every number *per se*, rather, we're coming up with a way to create every number, simply by starting with a number (0) and giving a procedure that will give us the "next" number ($n+1 = n \cup \{n\}$)... it is nasty, and I would like to explain more, but I neither know quite enough nor would want this to get in the way of equally exciting topics.

As a historical note: in the early 20th century, Bertrand Russell and Alfred North Whitehead wrote a multi-thousand-page book, *Principia Mathematica*, in which they attempted to derive **all** of mathematics, carefully and formally, from some starting logical axioms. (It was the careful reading of the *PM* that helped Kurt Gödel later devise his Incompleteness Theorems, in part to demonstrate the futility of such a project.) Anyway, one of the most famous lines from the book, which comes on page 379 (i.e., it took almost 400 pages to get to this point) is as follows[2]:

***54·43.** $\vdash :. \alpha, \beta \in 1 . \supset : \alpha \cap \beta = \Lambda . \equiv . \alpha \cup \beta \in 2$

Dem.

$\vdash . {*}54{\cdot}26 . \supset \vdash :. \alpha = \iota`x . \beta = \iota`y . \supset : \alpha \cup \beta \in 2 . \equiv . x \neq y .$

$[{*}51{\cdot}231] \qquad \equiv . \iota`x \cap \iota`y = \Lambda .$

$[{*}13{\cdot}12] \qquad \equiv . \alpha \cap \beta = \Lambda \qquad (1)$

$\vdash . (1) . {*}11{\cdot}11{\cdot}35 . \supset$

$\qquad \vdash :. (\exists x, y) . \alpha = \iota`x . \beta = \iota`y . \supset : \alpha \cup \beta \in 2 . \equiv . \alpha \cap \beta = \Lambda \qquad (2)$

$\vdash . (2) . {*}11{\cdot}54 . {*}52{\cdot}1 . \supset \vdash .$ Prop

From this proposition it will follow, when arithmetical addition has been defined, that $1 + 1 = 2$.

[2] http://en.wikipedia.org/wiki/File:Principia_Mathematica_theorem_54-43.png