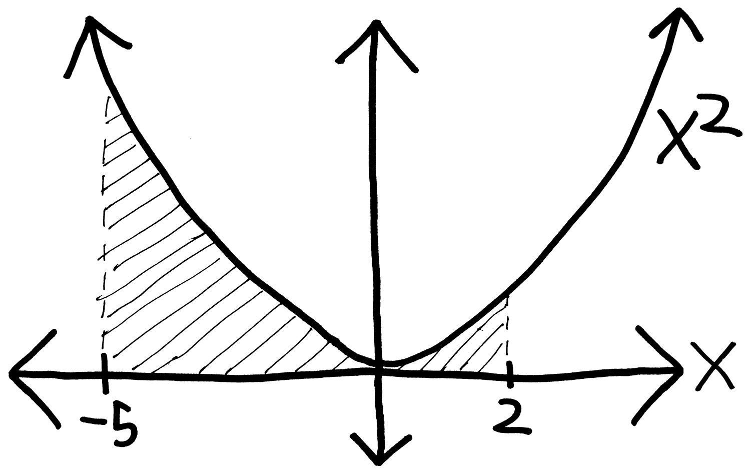
Improper Integrals

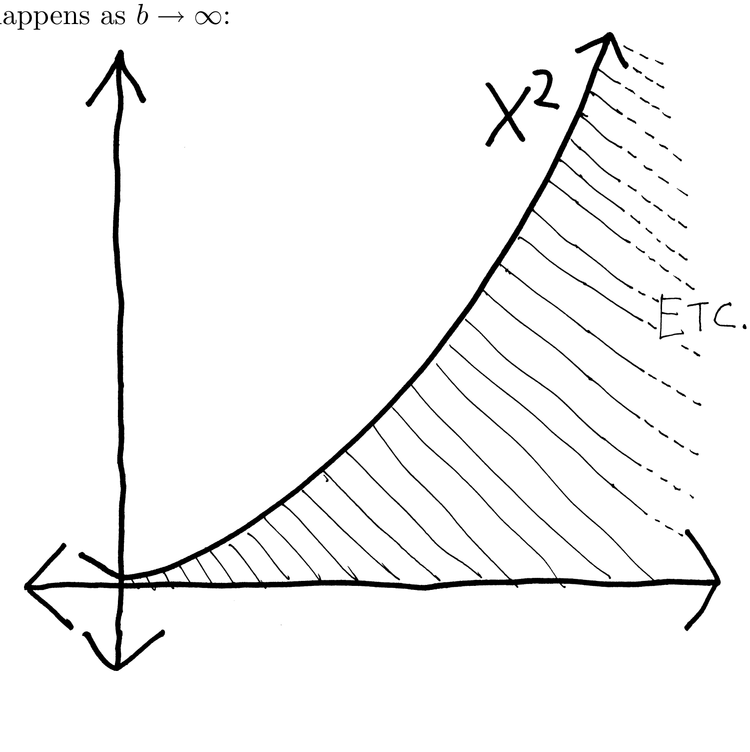
Calculus 12, Veritas Prep.

So far in our study of integrals—or, more specifically, in our study of the areas beneath curves—we've only considered areas of regions that start and end at definite points. We've learned how to calculate, for example, the area underneath x^2 from -5 to 2:



$$\begin{aligned}\int_{-5}^{2} x^2\,dx &= \left[\frac{1}{3}x^3\right]_{-5}^{2}\\ &= \frac{1}{3}(2)^3 - \frac{1}{3}(-5)^3\\ &= \frac{133}{3}\\ &\approx 44.3\end{aligned}$$

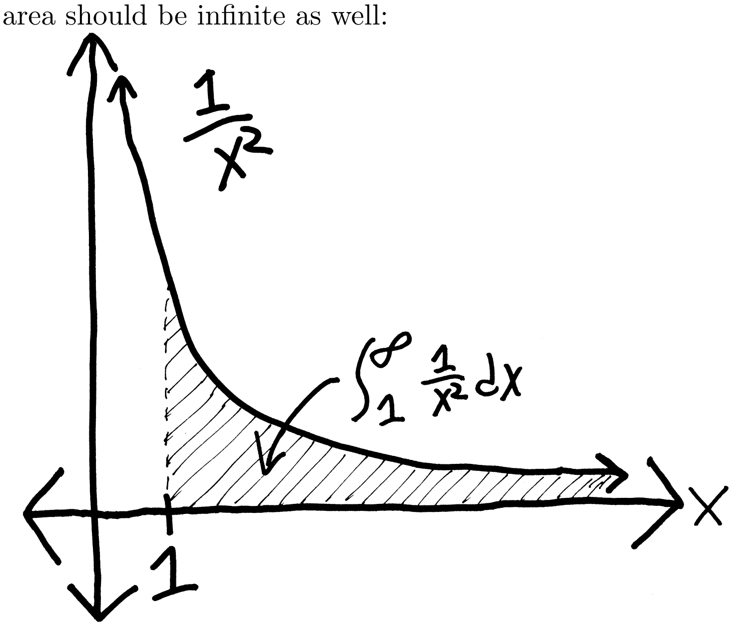
But what if we wanted to find... *all* the area underneath x^2? Like, from $-\infty$ to $+\infty$? Well, that's kind of a stupid question. The area is obviously infinite. And, indeed, if we did the math, that's what we'd find. We can't really plug ∞ into an integral, since it's not a real number *per se*, but we can use our limit magic—we can just plug some dummy variable in and then take the limit as that variable goes to ∞. So, for instance, to find the area beneath x^2 from 0 to ∞, we could take the integral from 0 to some number b, and then see what happens as $b \to \infty$:



$$\begin{aligned}\int_0^\infty x^2\,dx &= \lim_{b\to\infty}\int_0^b x^2\,dx\\ &= \lim_{b\to\infty}\left[\frac{1}{3}x^3\right]_0^b\\ &= \lim_{b\to\infty}\left(\frac{1}{3}b^3-\frac{1}{3}0^3\right)\\ &= \lim_{b\to\infty}\left(\frac{1}{3}b^3\right)\\ &= \infty\end{aligned}$$

Intuitively, this is obvious; formally, we confirm it.

Here's a slightly different example: what if we want to find the area beneath not x^2 between 0 and ∞, but what if we want to find the area beneath, say, $1/x^2$, between, say, 1 and ∞? This region looks slightly different—$1/x^2$ is asymptoting down to the x-axis rather than zooming up, like x^2—but the region is infinitely long, so its area should be infinite as well:

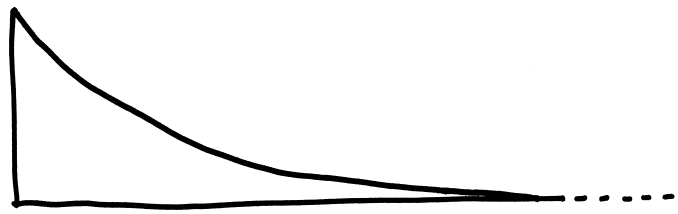


So if we do the math:

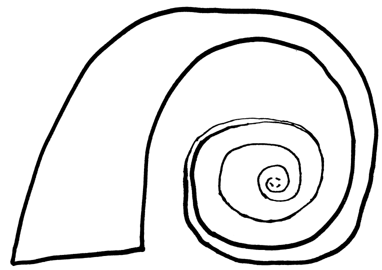
$$\begin{aligned}\int_0^\infty \frac{1}{x^2}\,dx &= \lim_{b\to\infty}\int_0^b \frac{1}{x^2}\,dx\\ &= \lim_{b\to\infty}\left[\frac{-1}{x}\right]_0^b\\ &= \lim_{b\to\infty}\left(\frac{-1}{b}-\frac{-1}{1}\right)\\ &= \lim_{b\to\infty}\left(\frac{-1}{b}+1\right)\\ &= \lim_{b\to\infty}\left(1-\frac{1}{b}\right)\\ &= 1\end{aligned}$$

... woah. The area isn't infinite. *The area is finite.* We have this shape that's *infinitely long* yet has *finite area.* This is bizarre. This is strange. What is going on???

One way of thinking about this is that this shape is a type of fractal. Normally $1/x^2$ from 1 to ∞ looks like this:

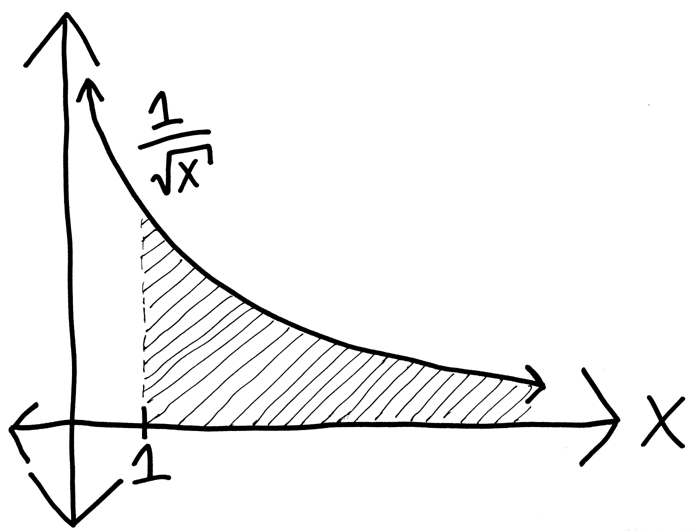


but I can roll it up into a spiral, a spiral that gets smaller and smaller and goes on forever, and yet which fits into a neat little circle of finite area:



AN INFINITE JELLYROLL! An infinitely-long jellyroll that ONLY HAS FINITE AREA! The real question is: where does one get a baking pan for such a jellyroll? (Hint: a certain Swedish department store's Geneva, Switzerland branch.)

Does this mean that every shape that's infinitely-long, but which is getting narrower, will have finite area? What if I have a different shape? What if I have, say, $1/\sqrt{x}$ from 1 to ∞? Like $1/x^2$, $1/\sqrt{x}$ also has a horizontal asymptote at $y = 0$. It's also getting closer and closer to the x-axis as x gets bigger and bigger:



But if we calculate the area beneath $1/\sqrt{x}$ betwen 1 and ∞....

$$\begin{aligned}\int_1^\infty \frac{1}{\sqrt{x}}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{\sqrt{x}}\,dx\\ &= \lim_{b\to\infty}\left[2\sqrt{x}\right]_1^b\\ &= \lim_{b\to\infty}\left(2\sqrt{b}-2\sqrt{1}\right)\\ &= \lim_{b\to\infty}\left(2\sqrt{b}-2\right)\\ &= \infty\end{aligned}$$

So NOW we have an area that's infinitely-wide, and also has infinite area!

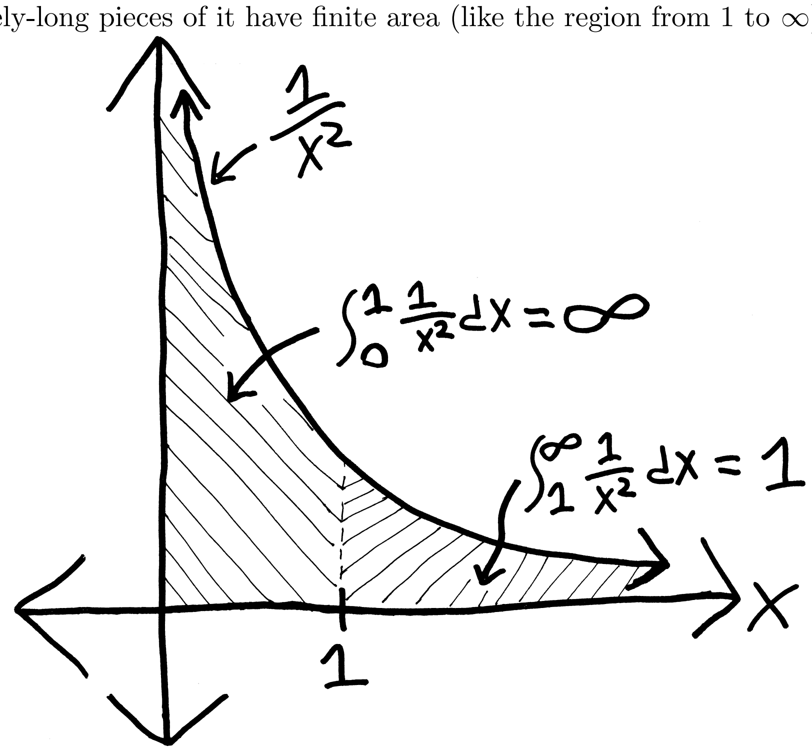
If you think about it, we deal with the infinite in two ways: we can have shapes that are infinitely *wide*, but we can also have shapes that are infinitely *tall*. We have horizontal asymptotes, and we have vertical asymptotes. What if, for example, we want to find $\int_0^1 \frac{1}{x^2}\,dx$? We know $1/x^2$ has a vertical asymptote at $x = 0$, so when we do the integral, we'll actually need to not do the integral from 0 to 1—we'd get some sort of divide-by-zero error—but do the integral from some dummy variable—say, a—to 1, and then take the limit as a approaches 0 from the right side:

$$\begin{aligned}
\int_0^1 \frac{1}{x^2}\,dx &= \lim_{a\to 0^+} \int_a^1 \frac{1}{x^2}\,dx \\
&= \lim_{a\to 0^+} \left[\frac{-1}{x}\right]_a^1 \\
&= \lim_{a\to 0^+} \left(\frac{-1}{1} - \frac{-1}{a}\right) \\
&= \lim_{a\to 0^+} \left(\frac{-1}{1} + \frac{1}{a}\right) \\
&= \lim_{a\to 0^+} \left(\frac{1}{a} - 1\right)
\end{aligned}$$

but as a approaches 0 from the right side (positive direction), $1/a$ gets bigger and bigger and bigger...

$$= \infty$$

So the area of this infinitely-tall shape is also infinite! But that's kind of weird—it means we have this one function, $1/x^2$, and some infinitely-long pieces of it have infinite area (like the region from 0 to 1), and other infinitely-long pieces of it have finite area (like the region from 1 to ∞):



Problems

Draw the functions and shapes described in the following integrals, and then find their areas:

1. $\int_1^\infty \frac{1}{x^2}\,dx$

2. $\int_5^\infty \frac{1}{x^2}\,dx$

3. $\int_1^\infty \frac{1}{x}\,dx$

4. $\int_5^\infty \frac{1}{x}\,dx$

5. $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$

6. $\displaystyle\int_1^\infty \frac{1}{\sqrt[3]{x}}\,dx$

7. $\displaystyle\int_1^\infty \frac{1}{x^4}\,dx$

8. $\displaystyle\int_1^\infty x\,dx$

9. $\displaystyle\int_1^\infty \frac{1}{\sqrt{1-x}}\,dx$

10. $\displaystyle\int_1^\infty \frac{\ln(x)}{x}\,dx$

11. $\displaystyle\int_1^\infty \frac{1}{x^4}\,dx$

12. $\displaystyle\int_{-\infty}^0 xe^x\,dx$

13. $\displaystyle\int_0^\infty e^{-x}\sin(x)\,dx$

14. $\displaystyle\int_0^\infty xe^{-x}\,dx$

15. $\displaystyle\int_0^\infty \sin(x)\,dx$

16. $\displaystyle\int_0^1 \frac{1}{x}\,dx$

17. $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$

18. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$

19. $\displaystyle\int_0^1 x\,dx$

20. $\displaystyle\int_0^1 x^5\,dx$

21. $\displaystyle\int_0^{\pi/2} \tan(x)\,dx$

22. $\displaystyle\int_{-\infty}^1 \frac{8x^3}{(x^4+1)^2}\,dx$

23. $\displaystyle\int_0^1 x\ln(x)\,dx$

24. $\displaystyle\int_0^2 \frac{1}{(x-2)}\,dx$

25. $\displaystyle\int_0^2 \frac{1}{(x-2)^2}\,dx$

26. $\displaystyle\int_0^2 \frac{1}{(x-2)^{2/3}}\,dx$

27. For what values of p is $\displaystyle\int_0^\infty x^p e^{-x}\,dx$ infinite? for what values of p is it finite?

28. Is $\displaystyle\int_1^\infty e^{-x^2}\,dx$ finite, or infinite? This is not particularly easy to answer, since you can't antidifferentiate e^{-x^2} without using an infinite series. So let me ask this: is $\displaystyle\int_1^\infty e^{-x}\,dx$ finite or infinite? Might knowing this aid you in knowing whether $\displaystyle\int_1^\infty e^{-x^2}\,dx$ is finite or infinite? (Note that I'm not asking for the precise area beneath e^{-x^2}, if it is finite; I'm merely curious as to whether it's finite or infinite.)

Improper Integrals, II

Calculus 12, Veritas Prep.

Name: ______________________________

Improper integrals are soooo cool. You can have a shape that's infinitely long but has finite area? How trippy is that? But, of course, sometimes you have a shape that's infinitely long that *does* have infinite area. Why the difference? Why can infinite shapes have finite area, anyway? Fill out this table, and see if you can start to probe this. (Obviously, you'll need to do the work on a separate sheet (or sheets) of paper.) **I think it might help if you drew the corresponding regions, too, rather than just calculated the integrals.**

k	x^k	$\int_1^\infty x^k\,dx$
3	x^3	
2	x^2	
1	x	
1/2	$x^{1/2} = \sqrt{x}$	
1/3	$x^{1/3} = \sqrt[3]{x}$	
1/4	$x^{1/4} = \sqrt[4]{x}$	
0	$x^0 = 1$	
$-1/4$	$x^{-1/4} = \frac{1}{\sqrt[4]{x}}$	
$-1/3$	$x^{-1/3} = \frac{1}{\sqrt[3]{x}}$	
$-1/2$	$x^{-1/2} = \frac{1}{\sqrt{x}}$	
-1	$x^{-1} = \frac{1}{x}$	
$-4/3$	$x^{-4/3} = \frac{1}{x^{4/3}}$	
$-3/2$	$x^{-3/2} = \frac{1}{x^{3/2}}$	
-2	$x^{-2} = \frac{1}{x^2}$	
-3	$x^{-3} = \frac{1}{x^3}$	
-4	$x^{-4} = \frac{1}{x^4}$	
-5	$x^{-5} = \frac{1}{x^5}$	

Suggestion: this looks like a huge number of integrals, but it's a lot less work than it seems. Many have the same answer. And it might make your life easier if you find the antiderivative of x^k first, and use that to evaluate all these integrals, rather than doing it anew in each case. But remember that doesn't work for $1/x$ (because you'll need to use a natural log instead).

You should see a pattern here. When k is less than -1, the area under x^k between 1 and ∞ is finite. When k is greater than (or equal to) -1, the corresponding area is infinite.

But that's just an empirical conjecture. It's just a guess, based on the evidence. Not a proof. So can you *prove* this to be true? My suggestion: consider the function $f(x) = x^k$, and try to find $\int_1^{\infty} x^k \, dx$. For convenience, you'll probably want to split it up into three seperate cases:

- when k is greater than -1,
- when $k = -1$, and
- when k is less than -1.

Write up your proof/argument on a seperate piece of paper and attach it to this. By "attach," I don't mean, "by folding it," or "by putting it nearby." I mean, "with a staple." If you don't own a stapler, buy one, and get your accountant to credit it as a tax write-off.

Next question: can you plot the value of the area under the function x^k between 1 and ∞ as a function of k? That is, can you plot the function $f(k) = \int_1^{\infty} x^k \, dx$? What does it look like? Do that, too, and attach it. When I was 19, I thought that graph was one of the most remarkable things I had ever seen.

Improper Integrals, III

Calculus 12, Veritas Prep.

Name: ___________________________

Yesterday, you analyzed the area beneath the functions x^k from 0 to ∞, and found that the area was infinite for $k \geq -1$ (sensibly enough), and finite for $k < -1$ (bizarrely). Today, I ask you to consider a related question: what is the area beneath x^k between 0 and 1? Sometimes this is obviously a finite number—for example, for x^2. The region between 0 and 1 is just a curvy-triangular shape. But sometimes (i.e., when $k < 0$), x^k has a vertical asymptote at $x = 0$—and so this region between 0 and 1 is infinitely tall! So the obvious question is: this infinitely-tall-region between 0 and 1—does it have infinite area, or finite area? when? Fill out this table, and see if you can start to probe this. (Obviously, you'll need to do the work on a separate sheet (or sheets) of paper.)

$\int_0^1 x^k\,dx$	x^k	$\int_1^\infty x^k\,dx$
	x^3	∞
	x^2	∞
	x	∞
	$x^{1/2} = \sqrt{x}$	∞
	$x^{1/3} = \sqrt[3]{x}$	∞
	$x^{1/4} = \sqrt[4]{x}$	∞
	$x^0 = 1$	∞
	$x^{-1/4} = \frac{1}{\sqrt[4]{x}}$	∞
	$x^{-1/3} = \frac{1}{\sqrt[3]{x}}$	∞
	$x^{-1/2} = \frac{1}{\sqrt{x}}$	∞
	$x^{-2/3} = \frac{1}{x^{2/3}}$	∞
	$x^{-3/4} = \frac{1}{x^{3/4}}$	∞
	$x^{-1} = \frac{1}{x}$	∞
	$x^{-4/3} = \frac{1}{x^{4/3}}$	3
	$x^{-3/2} = \frac{1}{x^{3/2}}$	2
	$x^{-2} = \frac{1}{x^2}$	1
	$x^{-3} = \frac{1}{x^3}$	1/2

	$x^{-4} = \frac{1}{x^4}$	1/3
	$x^{-5} = \frac{1}{x^5}$	1/4

Just like with last time, you should see a pattern here. Make a conjecture as to what's happening:

- when k is greater than -1, the area under x^k between 0 and 1 is:

- when k is equal to -1, the area under x^k between 0 and 1 is:

- when k is less than -1, the area under x^k between 0 and 1 is:

Now: can you *prove* your conjecture? Write up your proof/argument on a seperate piece of paper and staple it to this. Also: like last time, can you plot the value of the area under the function x^k between 0 and ∞ as a function of k? That is, can you plot the function $f(k) = \int_0^1 x^k \, dx$? What does it look like? Do that, too, and attach it.