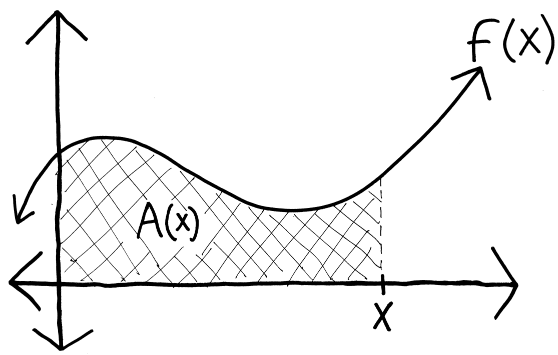
Integration

Calculus 11, Veritas Prep.

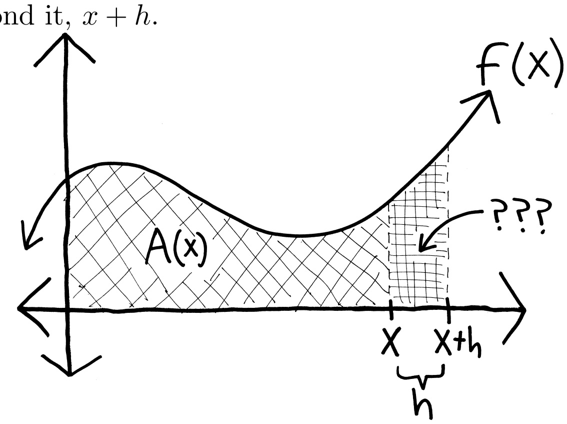
For all the formulas and the formalism, despite the rigors of the difference quotient and the difficulties of the quotient rule, calculus is about just two ideas: slopes and areas. These are two ideas that seem to have nothing in common with each other. And yet—as Newton and Leibniz discovered—these two operations are not only related, *they are the same operation.* Or, rather, they are inverses of each other: if we have a function, and we take its derivative, we get its slope. If we have a function and we take an antiderivative, we get its area. It makes sense that "slope" and "derivative" should be equivalent, since that was our very purpose in coming up with the formal idea of the derivative, but Newton and Leibniz's remarkable realization was that *area is just an antiderivative.*

Let me try to show you why. Imagine we have some function $f(x)$:

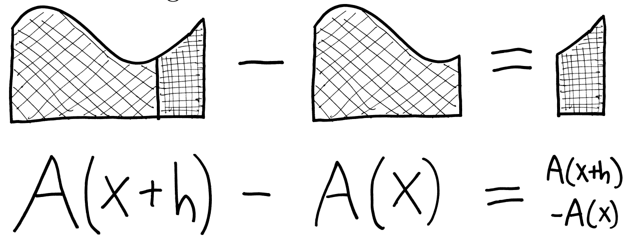
And imagine that the area beneath this function, from the origin out to some point x, is given by the function $A(x)$:



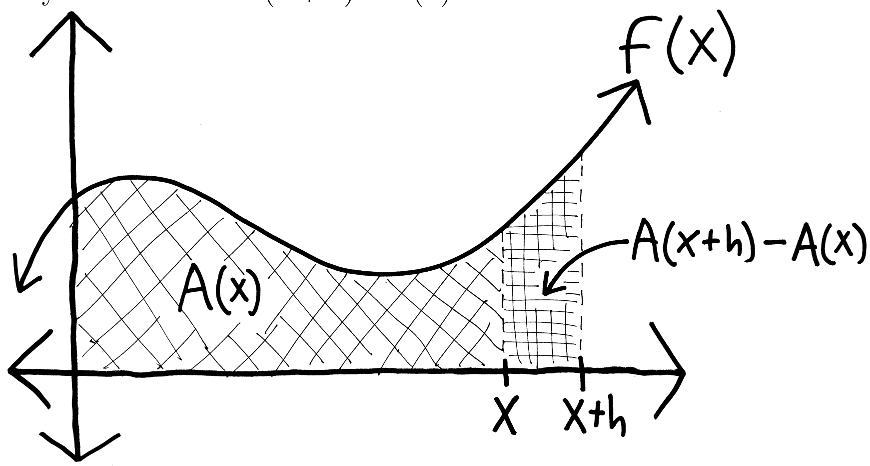
Obviously, we don't yet know what $A(x)$ is; our goal is to find some sort of formula for it. So bear with me. What if I want to find not the area underneath $f(x)$ from 0 to x, but just the area of a little sliver—from x to a point, say, h units beyond it, $x + h$.



We can actually come up with a formula for this using our function $A(x)$. We know that the area from 0 to x is $A(x)$. But x could be anything—it could be 5, it could be 6, it could even be $x + h$. So the area from 0 to $x+h$ must be $A(x+h)$. But we don't want to find the area from 0 to $x+h$—we want to find the area from x to $x+h$. So all I need to do is take the area from 0 to $x+h$, and subtract that big chunk I don't want—subtract the area of the region from 0 to x.

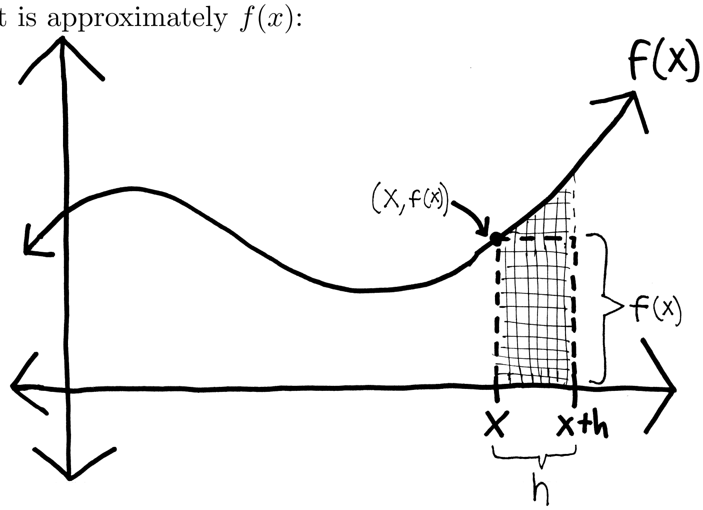


So then the area of my little sliver is $A(x+h) - A(x)$.



But there's another way that I can find this area. Namely: this is a little sliver, not a giant plank. h is pretty small. So then the region from x to $x+h$ is probably pretty close to a rectangle. Sure, there's that curvy bit at the top, but because h is reasonably small, that doesn't make that much of a difference.

But we already know how to find the area of a rectangle: width times height. The width of this little sliver is h, and the height is approximately $f(x)$:



So then the area must be roughly $f(x) \cdot h$. But then I have two different ways of writing the area of this sliver: the area is $A(x+h) - A(x)$, and the area is also (approximately) $f(x) \cdot h$. So I must have:

$$\text{approximate area} \approx \text{exact area}$$
$$f(x) \cdot h \approx A(x+h) - A(x)$$

Or just:

$$f(x) \approx \frac{A(x+h) - A(x)}{h}$$

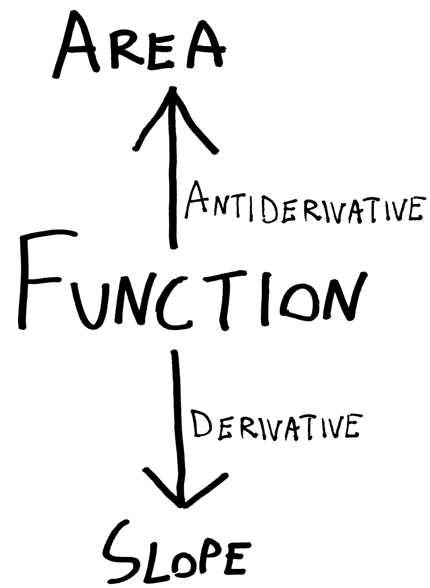
Moreover, as h gets smaller and smaller, this approximation gets better and better—as h goes to zero, these two things become *equal*:

$$f(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

(One of the things I dislike about writing, as opposed to teaching in person, is that it's harder to toy with the timing. Because what I really want you to do right now is stare at that equation and let it sink in and realize what just happened.)

This looks horrifyingly familiar. We tried to find the area of this shape—and we ended up with Fermat's difference quotient. What this is telling us is that if we take the derivative of this equation for the area, $A(x)$, we get the equation for the curve, $f(x)$. Conversely, if we had the equation for the curve, and took an antiderivative, we'd get the equation for the area. Or: *the area beneath a curve is just the antiderivative of the curve.*

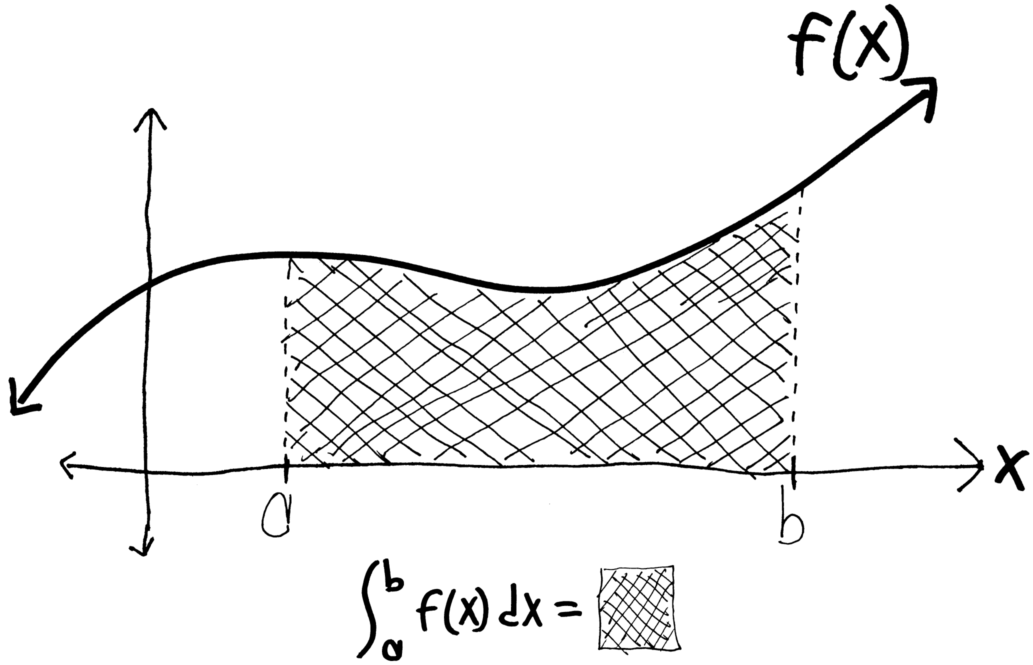
This is startling.



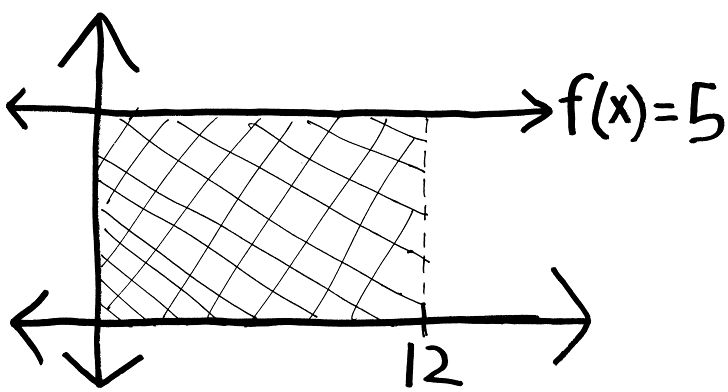
Somewhat More Formally

That was a somewhat informal exploration. Let's see if we can develop this idea further. First of all, let's call this function for the area an "integral," and define it thusly:

$$\int_a^b f(x)\,dx = \text{the area between } f(x) \text{ and the } x\text{-axis, between } a \text{ and } b$$



So, if we do this, we are really just creating a fancy notation for a very simple idea: the idea of area. One of the consequences of this is that there are already a bunch of integrals we already know. For instance, what if you want to find $\int_0^{12} 5\,dx$? You can do this without any fancy techniques! You know that the function 12 is just a horizontal line at $y = 12$, so if you want to find the area between $y = 5$, the x-axis, and the x-coordinates 0 and 12, all you need to do is find the area of a rectangle with width 12 and height 5:



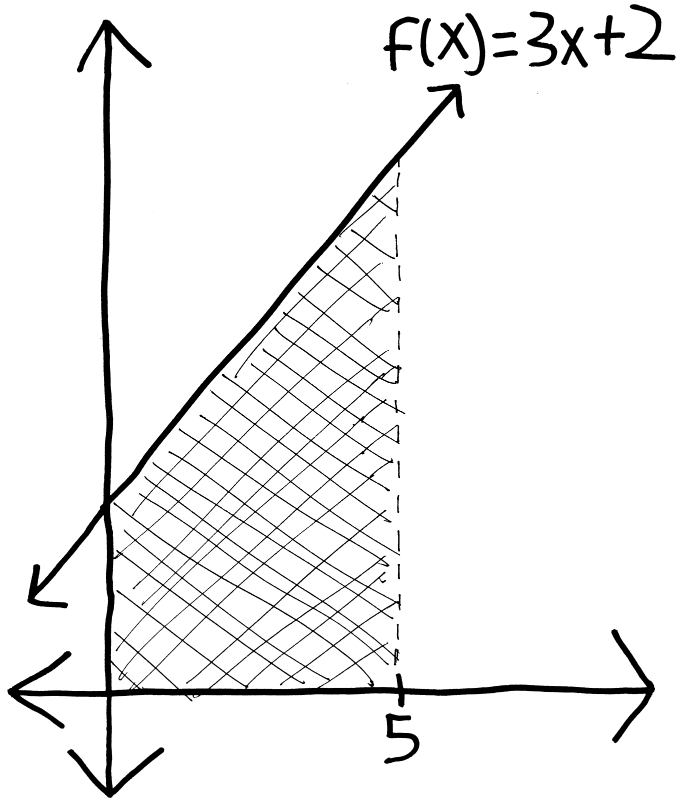
So we have:

$$\int_0^{12} 5\,dx = 12 \cdot 5 = 60$$

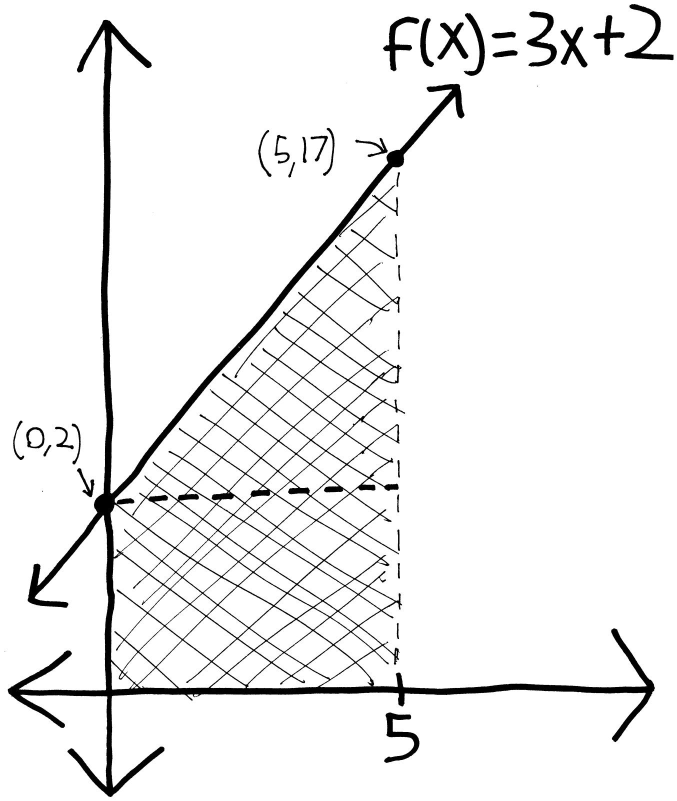
In the homework, I'll ask you to generalize this a little: what if you want to find the integral from a to b of the function $f(x) = k$?

This is not at all interesting; all we've done is apply a fancy symbolism to stuff you've known for years. You could go into an elementary school classroom, talk about "integrals" and use the $\int$ symbol, and even though you'd be doing absolutely nothing different, it'd seem completely incoherent and foreign and complicated... even though the idea is very simple.

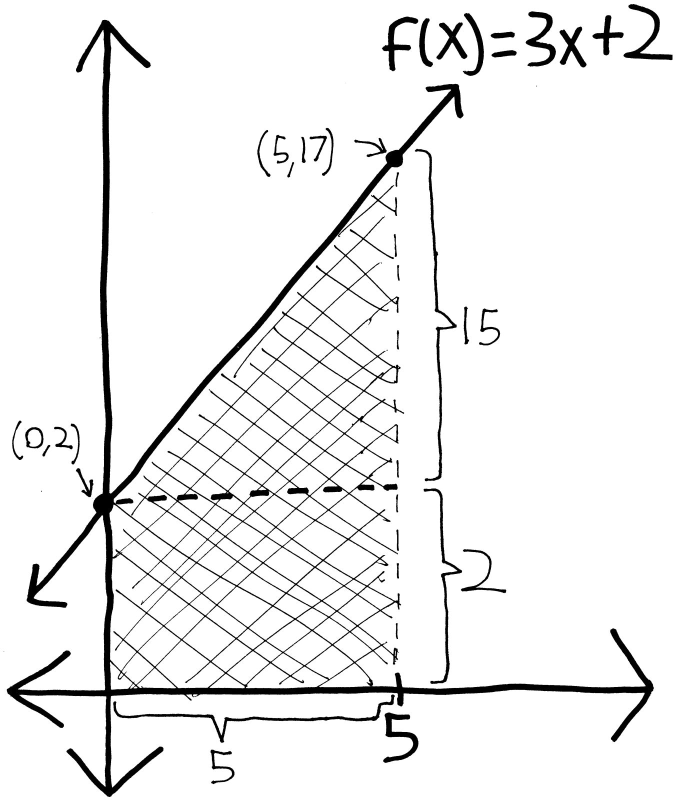
Here's another example: what if we have a sloped line, like $f(x) = 3x + 2$? What if we want to find $\int_0^5 3x + 2\,dx$?



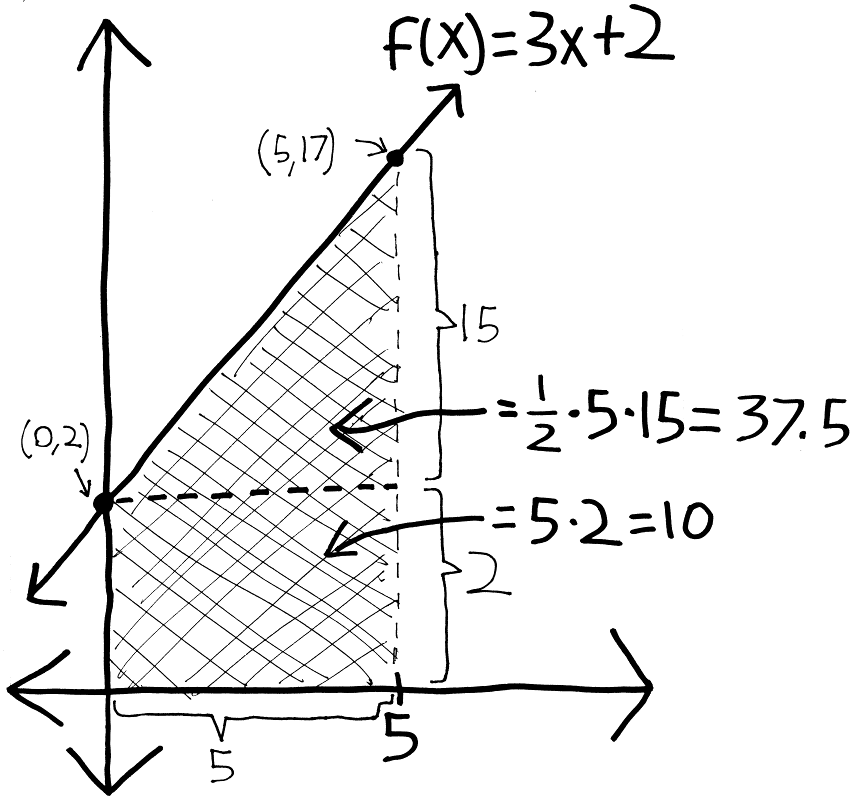
This is not hard, either—I just break it up into a triangle and a rectangle. (Or use a formula for the area of a trapezoid.) If I find the coordinates of all the relavent points, I have something like:



And then I can find the width and the height of each shape:



So I have a triangle with width 5 and height 17, so it has area $\frac{1}{2}5 \cdot 17 = 37.5$. And I have a rectangle with width 5 and height 2, so it has area $5 \cdot 2 = 10$



So then the total area of my shape is $37.5 + 10 = 47.5$. Put differently:

$$\int_0^5 3x + 2\,dx = 47.5$$

Problems

Using various non-fancy techniques, evaluate each of the following fancy expressions:

1. $\int_0^3 5\,dx$

2. $\int_0^b k\,dx$

3. $\int_a^b k\,dx$

4. $\int_5^{239} 7\,dx$

5. $\int_0^b x\,dx$

6. $\int_0^b x+6\,dx$

7. $\int_0^b x+\beta\,dx$

8. $\int_0^b \alpha x+\beta\,dx$

9. $\int_a^b \alpha x+\beta\,dx$

10. $\int_{-4}^4 \sqrt{16-x^2}\,dx$

11. $\int_{-r}^r \sqrt{r^2-x^2}\,dx$

12. $\int_0^r \sqrt{r^2-x^2}\,dx$

Towards a Unified Theory of Integration

There's a bit of an issue that we haven't addressed so far. In the previous section, we worked out simple integrals using formulas we already know for the areas of familiar shapes. What we've found is just a bunch of *ad hoc* methods for calculating integrals. We've said, more or less, that if the function is a straight line, we can find the integral by considering it as a rectangle; if the function looks like a triangle, we can use our formula for the area of a triangle; if the function looks like a trapezoid... all we've done is to make a correspondence between shapes of functions and shapes whose areas we already know how to find. We've made up this fancy symbolism that is pure decoration: it has zero utility. (Except possibly to confuse: why use words like "area" and letters like "A" when you could use words like "integral" and symbols like $\int f(x)\,dx$?)

Before that, we said that integrals are the same thing as areas, and we demonstrated that integrals are the same as antiderivatives by assuming that we had some function for the area beneath a curve. We assumed that this "take the area under $f(x)$ from 0 to x" function exists. We take this idea of "area" as a primitive, atomic idea, one that can't be further understood or broken down.

But... what *is* area, anyway? Does the concept of "area" make any sense? Well, that's kind of a stupid thing to say. Of course "area" makes sense! We've been talking about the areas of shapes our entire lives! "Area" IS a primitive concept! It makes perfect sense! Except... well, when we came up with the derivative, we had to spend a long time just talking about the *idea* of a derivative—we had to convince ourselves that this concept of the "slope" of a curvy line *does* make sense, that individual points can have slopes, and so forth. "Slope" isn't an idea that makes as much intuitive sense as area.

So if area is such an obvious concept, what is it? What is "area," anyway? Is it length times width? It can't be that, because that wouldn't work for a circle. Is it "the space something takes up" or "the space something takes up bounded by a perimeter" or "the two-dimensional space of a region bounded by a perimeter that a shape takes up"? Those "definitions" don't really give any clarity—they just use lots of fancy words to hide the fact that we can't really explain what area is.

Here's what I mean. When we talk about derivatives—well, a derivative is a slope, right? Except "slope" isn't really a concept of algebra or arithmetic. And all of our calculus is just algebra and arithmetic. It's just a fancy set of ideas layered on top of algebra and arithmetic, a beautiful piece of architecture constructed out of these previous mathematical concepts. But "slope" isn't really a primitive concept of algebra and arithmetic. Not in the way that things like "plus" and "five" and "equals" are. So when we built the derivative, we had to come up with Fermat's Difference Quotient as a way of *translating* our intuitive idea of "slope" into the language of arithmetic and algebra.

It seems to be the same for area. "Area" isn't a basic concept of algebra or arithmetic. But we seem to have some intuitive idea of what "area" is. So can we translate that into arithmetic and algebra?

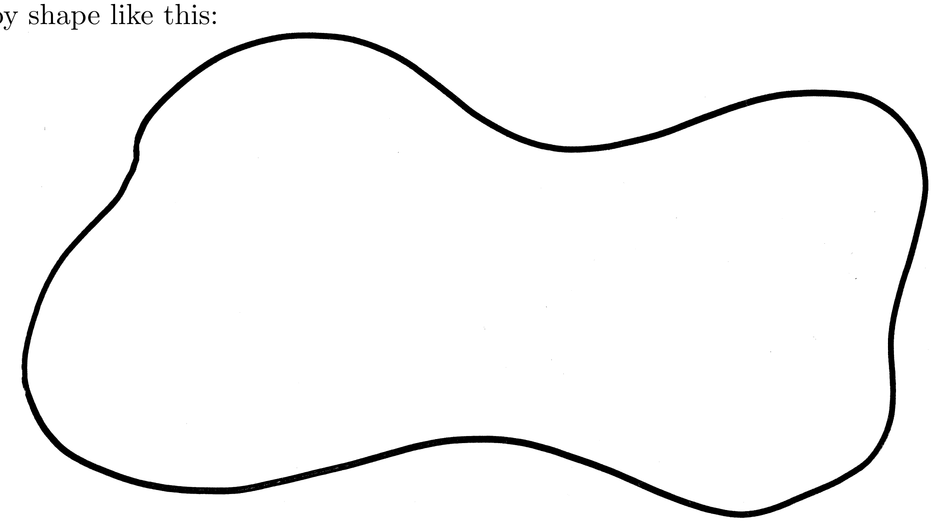
We know a bunch of specific areas:

- Area of a circle $= \pi r^2$
- Area of a rectangle $=$ width $\cdot$ length

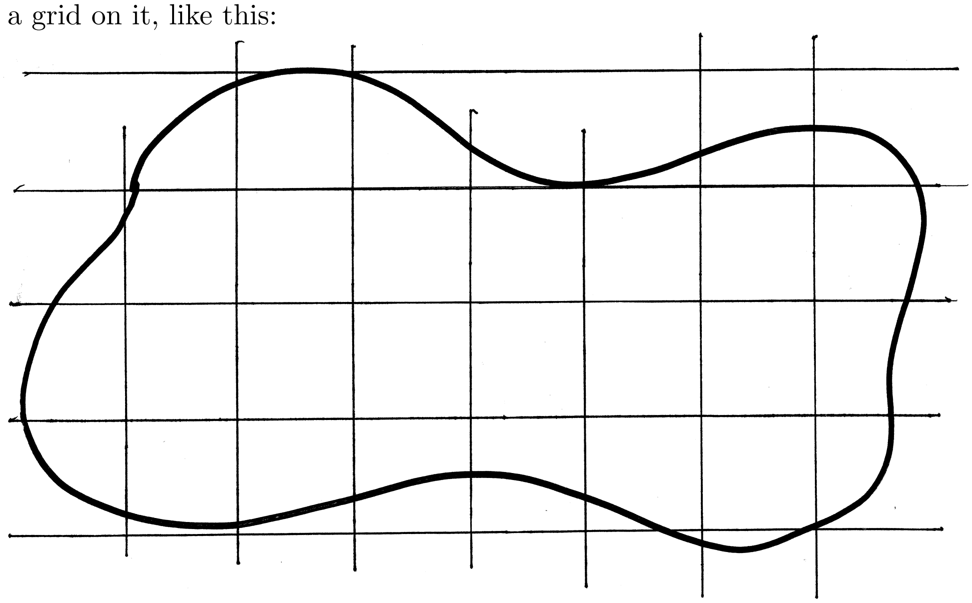
- Area of a triangle $= \frac{1}{2}\text{base} \cdot \text{height}$

But surely we don't want to define area on a case-by-case basis! Surely we don't want to say, "Well, if you have this shape, the area is this; if you have this shape, the area is this," and so forth. Surely there is some fundamental, singular concept of AREA beneath all of these specific formulas! Can we come up with a single, overarching definition of an area? One method that will always work? One rule to win every time? As the heirs to Aristotle, Aquinas, and Russell, can we at last realize this Platonic dream of fully ordering the universe?

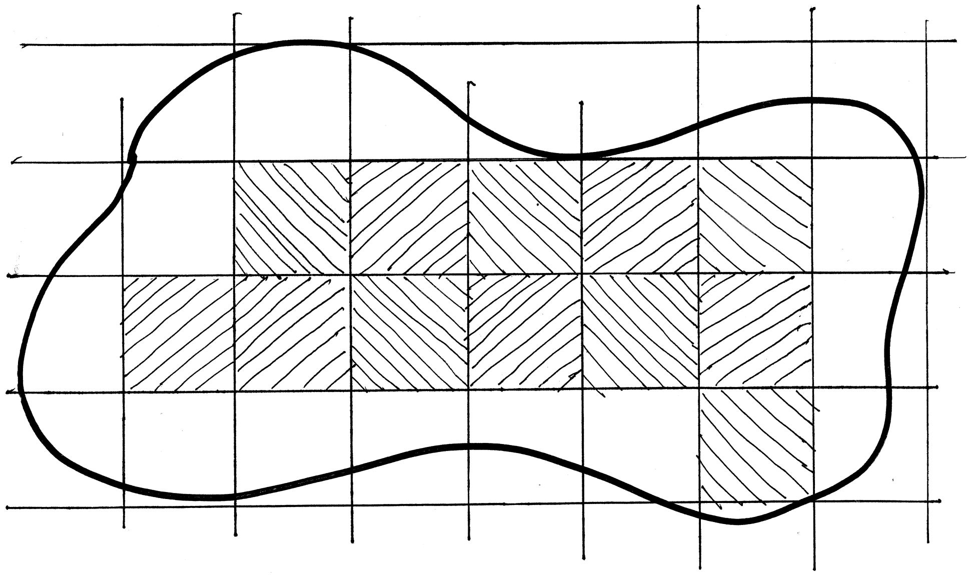
Allow me to make a suggestion. First of all, we don't have to come up with an equation *per se*, at least not yet. We could come up with a procedure (an algorithm) instead. So what if we did this: what if we take a blobby shape like this:



And draw a grid on it, like this:



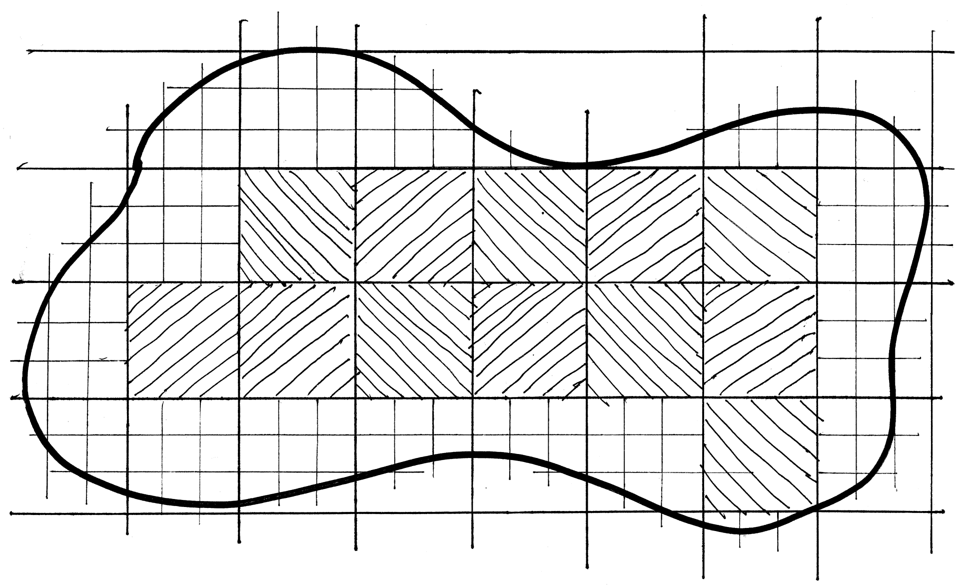
And then fill in all the boxes that are completely inside of the shape:



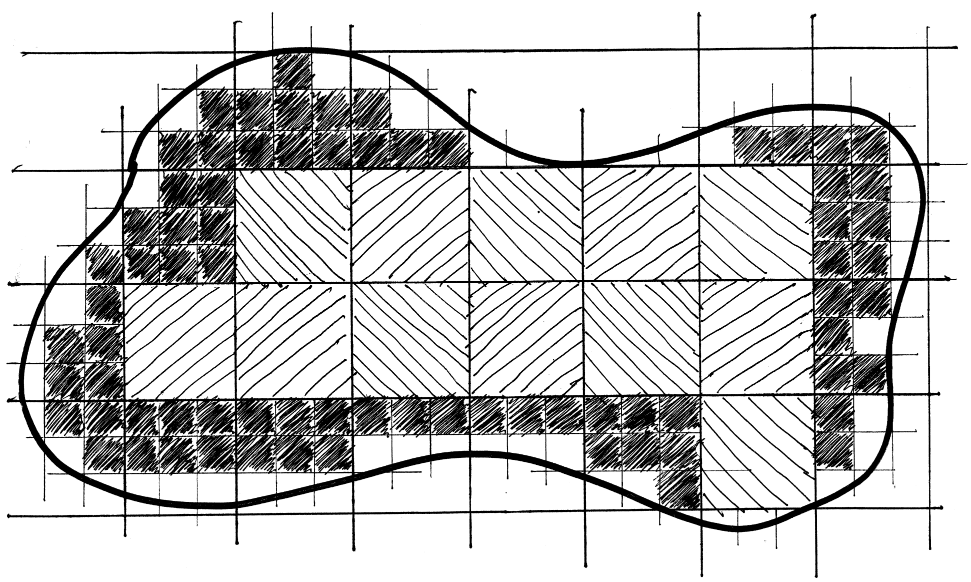
Then we could say that the area of this blobular shape is 12 boxes.

$$\text{Area} = 12 \text{ boxes}$$

Of course, that's only an approximation, since we still have all of those boxes that are partly inside the shape and partly outside. So what if we repeat this procedure, and draw a SMALLER grid inside each of those boxes, and then color in the littler boxes? If we draw a grid that splits each box up into 9 sub-boxes:



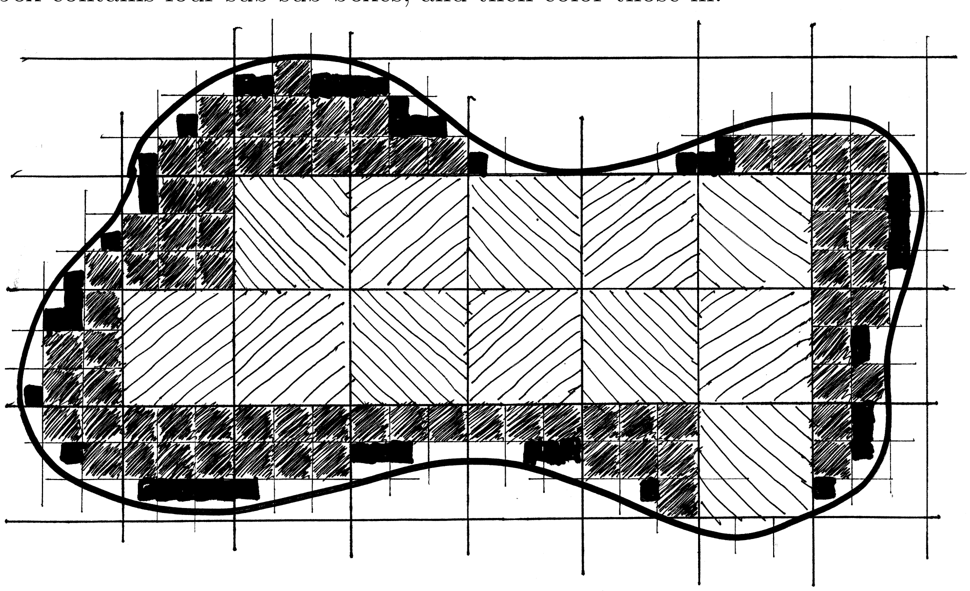
And then fill in the sub-boxes:



We can fill in 72 of the little sub-boxes. Each sub-box is 1/9 of a full box, so the total area, then, of our shape must be $12 + 72/9$ boxes.

$$\begin{aligned} \text{Area } &= 12 + \frac{72}{9} \text{ boxes} \\ &= 20 \text{ boxes} \end{aligned}$$

But this is STILL just an approximation, because we still have sub-boxes that are partially inside and partially outside of the shape. So let's repeat our procedure: let's split each sub-box up into a grid, so that each sub-box contains four sub-sub-boxes, and then color those in:



$$\begin{aligned} \text{Area } &= 12 + \frac{72}{9} + \frac{50}{36} \text{ boxes} \\ &\approx 21.389 \text{ boxes} \end{aligned}$$

And we can just repeat this procedure, on and on, *ad infinitum.* And so let us define the area in this way: area is what the number of boxes approaches as we repeat this procedure more and more times. If we define area like this, our definition will work for any shape—triangle, circle, rectangle, blob—and, presumably, it will give us the same answer as our intuitive concept of "area" does.

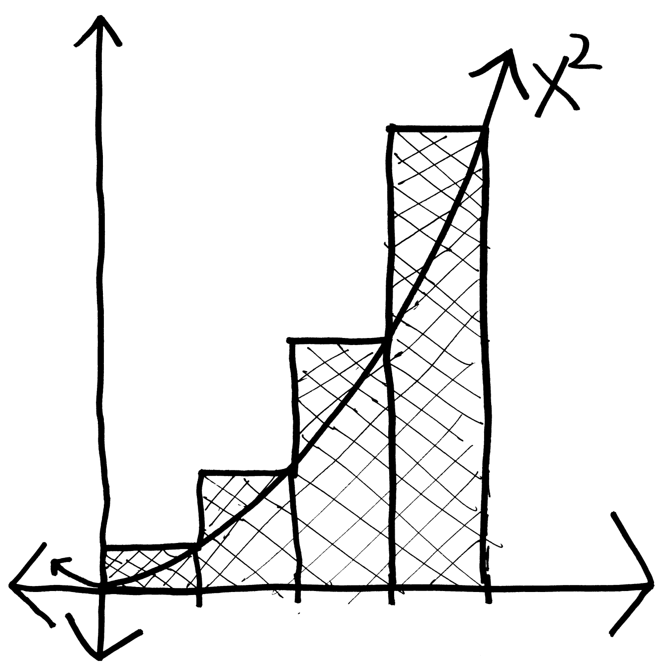
With Functions

Of course, our project is slightly more specific: we want to find the area of not just any shape, but the area beneath a function. So allow me to suggest a slightly simplified version of this "split it up into boxes" definition/procedure for finding area.

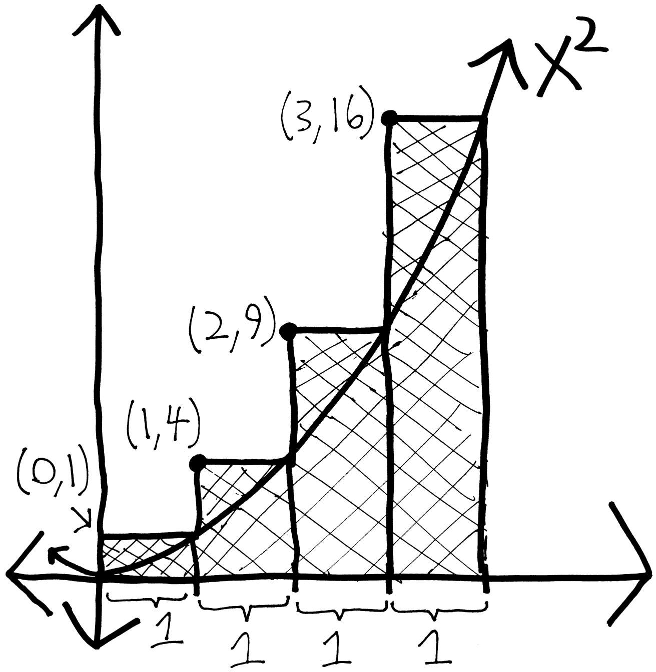
Allow me to suggest this as a procedure: rectangles. When we came up with our super-Pythagorean theorem, we constructed it by using the old Pythagorean theorem. When we came up with our formula for the slope of a curvy line (the derivative), we constructed it using the slope of a straight line. We were able to build on our particular knowledge in order to understand the general case. Let's do that here. What's the simplest shape to find the area of? A rectangle! Its area is just base times height! So—in the same way that in constructing the derivative we considered every function to consist of an infinite number of infinitely-short straight lines—what if we think of every region beneath a function to consist of an infinite number of infinitely-narrow rectangles? Then we could just add up the area of all these rectangles, and we'd have the total area. (An infinitely-large number of infinitely-small things! The continued clash of infinities!!!)

This is a rather ambitious project, so let's start somewhere simpler. What if we want to just approximate the area underneath a curve? We could do it with a finite number of rectangles. We could just make some boxes, fit them beneath our curve, and then add up their areas.

For instance, let's say I want to take a gander at what the area beneath x^2 from 0 to 4 is. I'll split the area into four boxes, each one unit wide, and I'll draw the boxes so that their tops hit x^2 on the right side, like so:



So then, if I work out the heights of the boxes, I get something like:

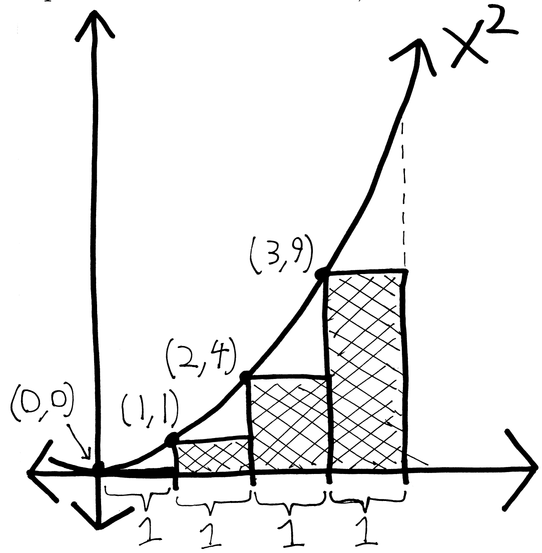


And so, all told I'll get, as my area:

$$\begin{aligned}\text{area underneath } x^2 \text{ from 0 to 4} &\approx 1{\cdot}1 + 1{\cdot}4 + 1{\cdot}9 + 1{\cdot}16 \\ &\approx 30\end{aligned}$$

Obviously this isn't the exact area[1], but it's probably pretty close. If anything, our estimate is probably a slight overestimate, since we've got those extra curvy-triangles of area on top.

So maybe I could try again, but instead of drawing the boxes such that their tops hit x^2 on the right side, I could draw them so their tops hit x^2 on the left side, like so:



And then I get

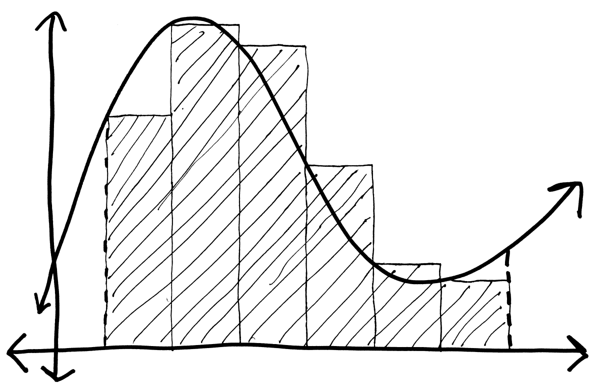
$$\begin{aligned}\text{area underneath } x^2 \text{ from 0 to 4} &\approx 1{\cdot}0 + 1{\cdot}1 + 1{\cdot}4 + 1{\cdot}9 \\ &\approx 15\end{aligned}$$

So this different approximation tells me that the area is about 15. I guess I can draw a couple of conclusions:

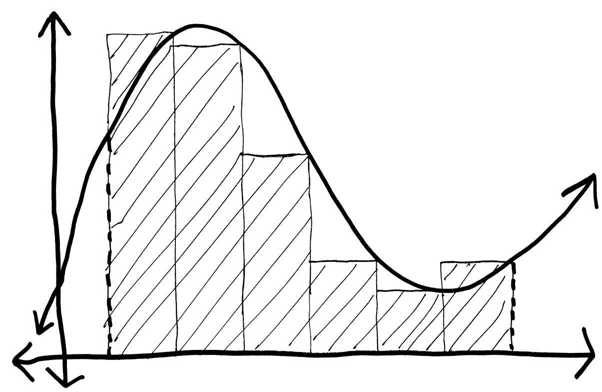
[1]because the exact area is 21 1/3.

1. This second approximation is an underestimate, since my boxes are all slightly below x^2. Thus, the actual area is probably somewhere between 15 and 30.
2. That's a big range for an estimate—15 and 30 differ by a factor of two. It would be nice to have a better idea of the actual area. One way to do this would be to use more boxes! Instead of 4, why not 5? or $5,000,000$? Or ∞?

Quick vocabulary note: this type of procedure is called a **Riemann sum**, after Bernhard Riemann (1826–1886). Riemann sums are the Legos that we will construct our integral out of—we will (in a minute) formally define the integral as being an infinite Riemann sum. There are all sorts of different ways we could make Riemann sums. We could make them (as in the previous example) with boxes of the same width, with the heights drawn on the left or right sides:

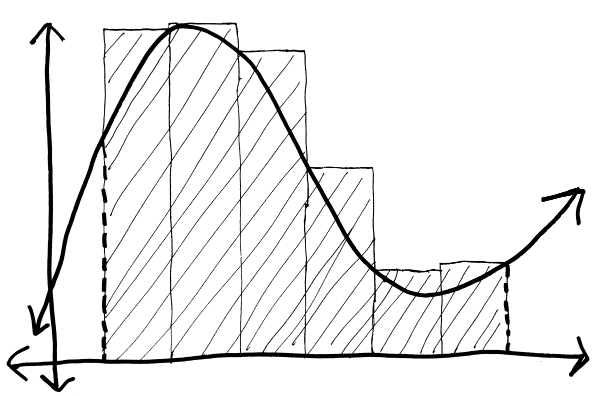


A Riemann sum with six equal-width partitions, with box heights drawn on the left endpoints

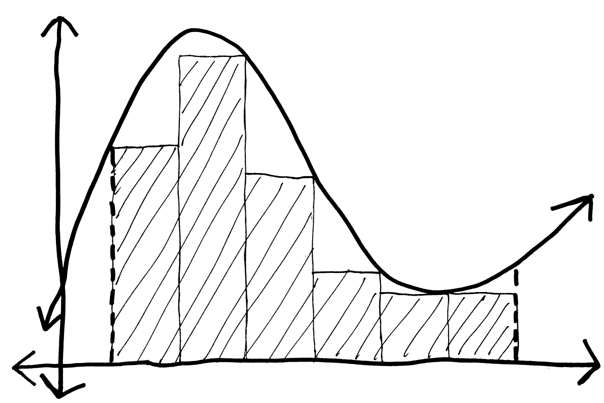


A Riemann sum with six equal-width partitions, with box heights drawn on the right endpoints

Or we could make them with the heights drawn in the middle, or three-quarters of the way to the left, or with the heights drawn *anywhere* within the box. I could draw the heights at the highest point in the box, or the lowest point (known, respectively, as an **upper sum** and a **lower sum**):

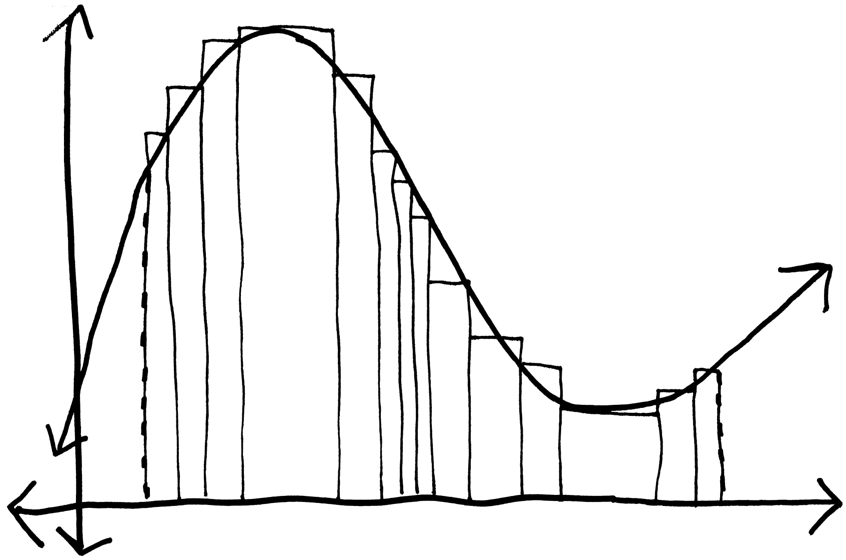


An upper sum with six partitions



A lower sum with six partitions

(These types of Riemann sums—upper and lower sums—will be crucial in our proof of the Fundamental Theorem of Calculus.) Or I could make a bunch of boxes of random width, with the height drawn at some random point:



The point is, *it doesn't actually matter* how we draw our boxes, because **the more boxes we have, the closer the Riemann sum will be to the actual area.** Always. As the number of boxes goes to ∞, the Riemann sum approximation will approach the actual area.

Problems

For each of the following problems, estimate the given area using each of the following methods. For each estimation, sketch the function and the relevant Riemann boxes. (I know these questions are tedious, but do them anyway.)

(a) A Riemann sum with three partitions, and box heights drawn on the left endpoints,
(b) a Riemann sum with three partitions, and box heights drawn on the right endpoints,
(c) a Riemann sum with three partitions, and box heights drawn in the center of each box,
(d) a Riemann sum with six partitions, and box heights drawn on the left endpoints,
(e) a Riemann sum with six partitions, and box heights drawn on the right endpoints,
(f) a Riemann sum with six partitions, and box heights drawn in the center of each box,
(g) an upper sum with three partitions,
(h) a lower sum with three partitions,
(i) an upper sum with six partitions,
(j) a lower sum with six partitions,
(k) a Riemann sum with any number of partitions and box heights chosen in absolutely any way you want,
(l) and by asking a younger sibling. (If you don't have a younger sibling, ask a friend if you can borrow theirs.)

134. The area underneath the function $f(x) = x^2$ between 0 and 3.

135. The area underneath $f(x) = x^3 + 8$ between -2 and 4.

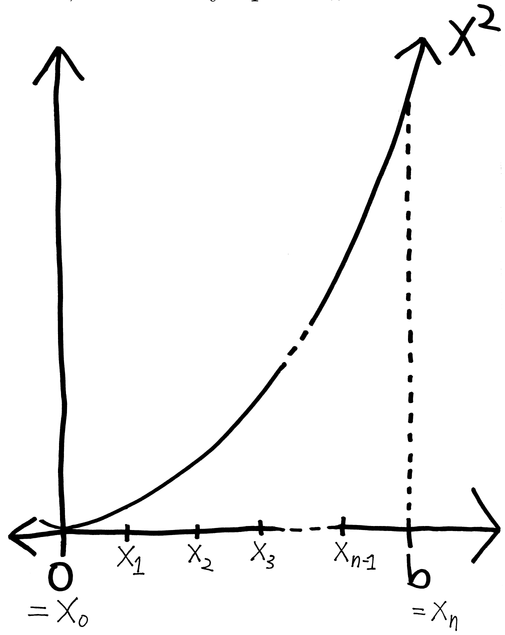
136. The area underneath the function $f(x) = e^x$ between 0 and 6.

137. The area underneath the function $f(x) = \sqrt{x}$ between 2 and 8.

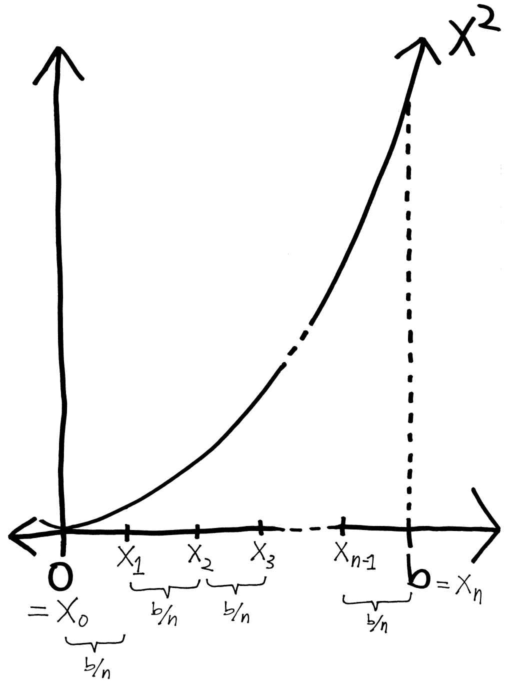
Infinite Riemann Sums

We've done Riemann sums with a finite number of boxes. But you know what'd be awesome? An INFINITE number of boxes, all INFINITELY narrow! Then we'd find not just the approximate area—but the EXACT area!!!

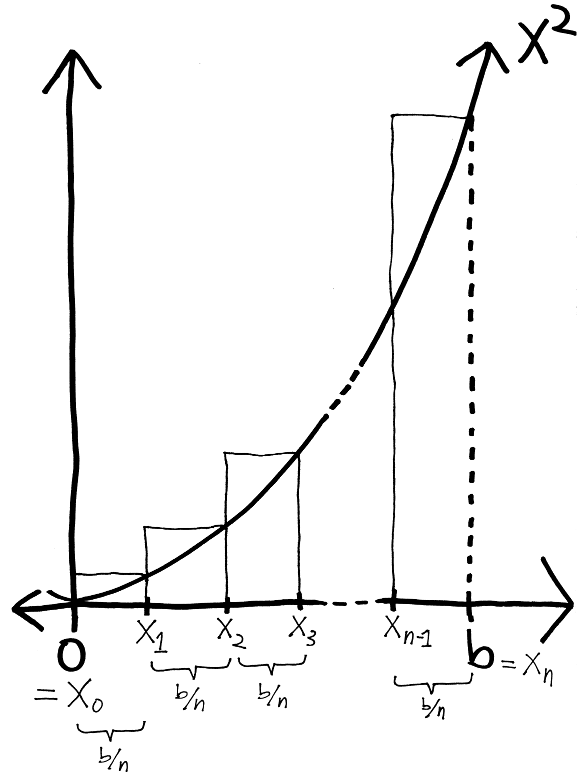
Let's do this with our old curvy friend, x^2. What if I want to estimate the area underneath x^2 (from, say 0 out to some number b) using a Riemann sum? And what if I want to do this not using a Riemann sum with a finite number of boxes, but with an infinite number? I guess that in order to do this, I'd need to involve a limit somehow—I'd have to come up with a formula for a Riemann sum of x^2 with n boxes, and then take a limit as $n \to \infty$. So let's see if we can do this. Let's imagine that I cut my area into boxes at the points x_0, x_1, x_2, x_3, and so on, all the way up to x_n.



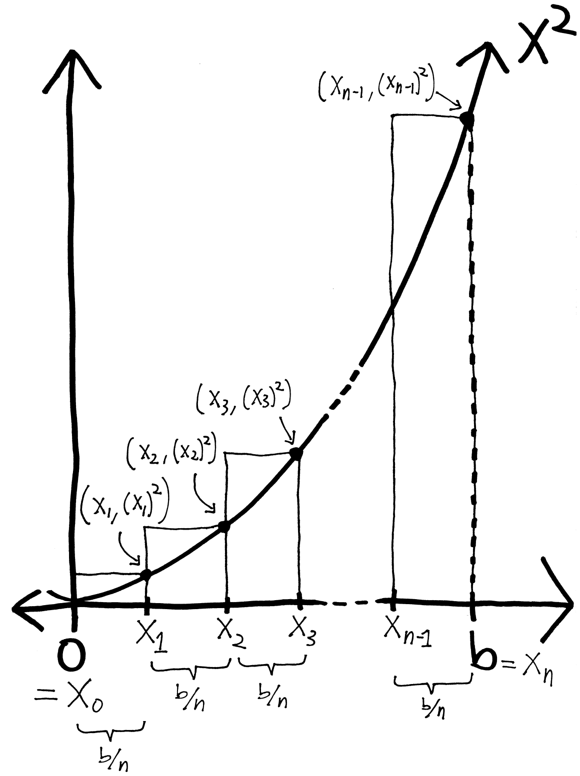
Note, incidentally, that because my area goes from 0 to b, $x_0 = 0$ and $x_n = b$. Also, for convenience, let's assume that each of these boxes are the same width. In that case, we know that, since there are n boxes fitting into the space from 0 to b—a space b units wide—each box must be b/n wide.



We don't have boxes yet—we just have the bases of boxes. (The foundations of skyscrapers, but no superstructure yet!) So let's draw these boxes. We could, of course, draw their heights anywhere; for no particular reason[2] let's draw them with their heights on the right side:



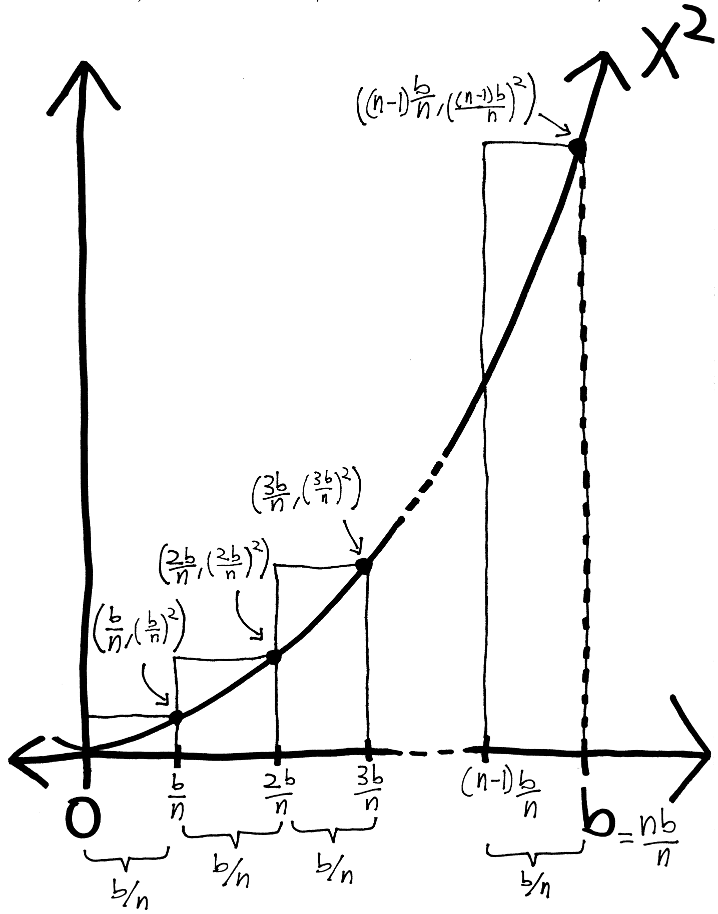
So let's see if we can figure out the heights of each of these boxes. This function is x^2, so to find the height, all I need to do is square the right-hand coordinate of the box. For instance, the first box must have height $(x_1)^2$, the second box must have height $(x_2)^2$, and so on.



Except we can simplify this a bit. We already know that each box is b/n wide. But this means that x_1

[2]other than that I've done this problem before, and this choice makes the algebra work out easiest

(the right-hand side of the first box) must be at b/n. x_2 must be at $2 \cdot b/n$. x_3 must be $3 \cdot b/n$. And so on.



- So really, the height of the first box is: $(x_1)^2 = \left(\frac{b}{n}\right)^2 = \frac{b^2}{n^2}$

- The height of the second box is: $(x_1)^2 = \left(\frac{2 \cdot b}{n}\right)^2 = \frac{2^2 b^2}{n^2}$

- The height of the third box is: $(x_1)^2 = \left(\frac{3 \cdot b}{n}\right)^2 = \frac{3^2 b^2}{n^2}$

- And so forth. Put differently, the height of the kth box will be: $\frac{k^2 b^2}{n^2}$

Moreover, the width of the kth box will be b/n (all the boxes are b/n wide). So then the area of the kth box will be:

$$\begin{aligned} \text{area of } k\text{th box} &= \text{width} \cdot \text{height} \\ &= \frac{b}{n} \cdot \frac{k^2 b^2}{n^2} \\ &= \frac{k^2 b^3}{n^3} \end{aligned}$$

So if I want to find the area of all these n boxes added together, I have something like:

$$\text{area of all the boxes } = \sum_{k=1}^{k=n} \frac{k^2 b^3}{n^3}$$

and since b and n are constant w.r.t. k, I can pull them out of the sum:

$$= \frac{b^3}{n^3} \sum_{k=1}^{k=n} k^2$$

So this gives us a formula for the area of a Riemann sum with n boxes, beneath x^2 from 0 to b. However, to turn this into not just an approximation of the exact area but the exact area itself, we'll need to take a limit as $n \to \infty$.

$$\text{exact area} = \lim_{n\to\infty} \left[\frac{b^3}{n^3} \sum_{k=1}^{k=n} k^2 \right]$$

But we still have this issue of the $\sum k^2$. This isn't a particularly satisfying answer, since we haven't worked out the limit. And it's not clear how to deal with the $\sum k^2$. Presumably, were we to work out that sum somehow, there'd be an n in it (it's the sum from $k = 1$ to $k = n$; it's the sum of the first n squares). So allow me to suggest that we refer back to a fun formula from algebra. You may or may not remember this, but if you want to add up the first n squares ($1^2 + 2^2 + 3^2 + 4^2 + \cdots$), there's an easy formula:

$$\sum_{k=1}^{k=n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

So if we plug that in here, we get that the area of this n-boxed Riemann sum is:

$$\begin{aligned}
\frac{b^3}{n^3} \sum_{k=1}^{k=n} k^2 &= \frac{b^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \\
&= \frac{b^3}{n^3} \cdot \left(\frac{2n^3 + 3n^2 + n}{6} \right) \\
\\
&= \frac{b^3}{n^3} \cdot \left(\frac{2n^3}{6} + \frac{3n^2}{6} + \frac{n}{6} \right) \\
&= b^3 \cdot \left(\frac{2n^3}{6n^3} + \frac{3n^2}{6n^3} + \frac{n}{6n^3} \right) \\
&= b^3 \cdot \left(\frac{2}{6} + \frac{3}{6n} + \frac{1}{6n^2} \right) \\
&= b^3 \cdot \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} \right)
\end{aligned}$$

And to find the exact area, we take a limit as $n \to \infty$:

$$\begin{aligned}
\lim_{n\to\infty} \left[b^3 \cdot \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} \right) \right] &= b^3 \lim_{n\to\infty} \left[\frac{1}{2} + \frac{1}{2n} + \frac{1}{6n^2} \right] \\
&= b^3 \cdot \frac{1}{3} \\
&= \frac{1}{3} b^3
\end{aligned}$$

So the area underneath x^2 from 0 to b is just $\frac{1}{3}b^3$! Put differently:

$$\int_0^b x^2 \, dx = \frac{1}{3} b^3$$

This is interesting and useful. **It is useful** because, for example, in our previous problem of finding the area beneath x^2 from 0 to 4, we could just use this formula to find the exact area. In that case $b = 4$, so as the exact area, we'd have:

$$\frac{1}{3}b^3 = \frac{1}{3}4^3 = \frac{1}{3}64 = 21 + \frac{1}{3}$$

It is interesting, because $\frac{1}{3}b^3$ looks an awful lot like the antiderivative of x^2. I mean, it has a b instead of an x, but still...

Problems

1. We now have a formula for $\int_0^b x^2\,dx$. But what's $\int_a^b x^2\,dx$? (Think about how the geometry works.)

2. Using a method analogous to the one above, find the exact area underneath the functions x^3 and x^4 from 0 out to b. That is, evaluate $\int_0^b x^3\,dx$ and $\int_0^b x^4\,dx$. The following formulas may help you:

$$\sum_{k=1}^{k=n} k^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 \quad \text{and} \quad \sum_{k=1}^{k=n} k^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$$

3. As in #1, use problem #2 and geometry to find formulas for $\int_a^b x^3\,dx$ and $\int_a^b x^4\,dx$.

The Integral, Formally

So, earlier we informally defined an integral as being the "area" between a curve and the x-axis. But what's an area? Does algebra know about areas? Does calculus know about areas? We might have asked the same question about slopes and derivatives—we wanted to come up with an equation for the slope of a function, but "slope" is not a concept of algebra and arithmetic (not in the same way that "addition" or "five" are). So instead we defined the derivative formally as $\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$, with the understanding that this captures, algebraically, our intuitive concept of "slope." We want to do something similar for an integral: using only our well-defined concepts of arithmetic, algebra, and calculus, we want to come up with an equation that is more-or-less equivalent to our intuitive notion of "area"[3].

Now that we have added Riemann sums to our arsenal, we can define an integral[4] formally. Here we

[3]Of course, one of the issues here is that intuitive concepts are, by definition, not particularly well-defined, and so formalizing them is, to some degree, subjective. This is the perennial issue in the quantitative social sciences: if I want to measure something (GDP, happiness, etc.), how do I measure it? what quantitative metric best captures my intuitive understanding of the concept? what if my intuitive understanding is not exactly the same as everyone else's? Pick a subject at random and follow the debates: the arguments are overwhelmingly about context rather than content: "I think that's a bad way of measuring economic inequality," not "I think you made those numbers up." Wittgenstein: "The existence of the experimental method makes us think we have the means of solving the problems which trouble us; though problem and method pass one another by" (*PI* 2.XIV).

[4]or, at least, a *Riemann* integral—one of the trippiest things about integration is that there are all sorts of different integrals that one can define, some of which work in some situations but not others. The usual next step, if you're taking a "real analysis" class, is to talk about **measure theory** so that you can define and understand the **Lebesque integral** (pronounced "la-bayg").

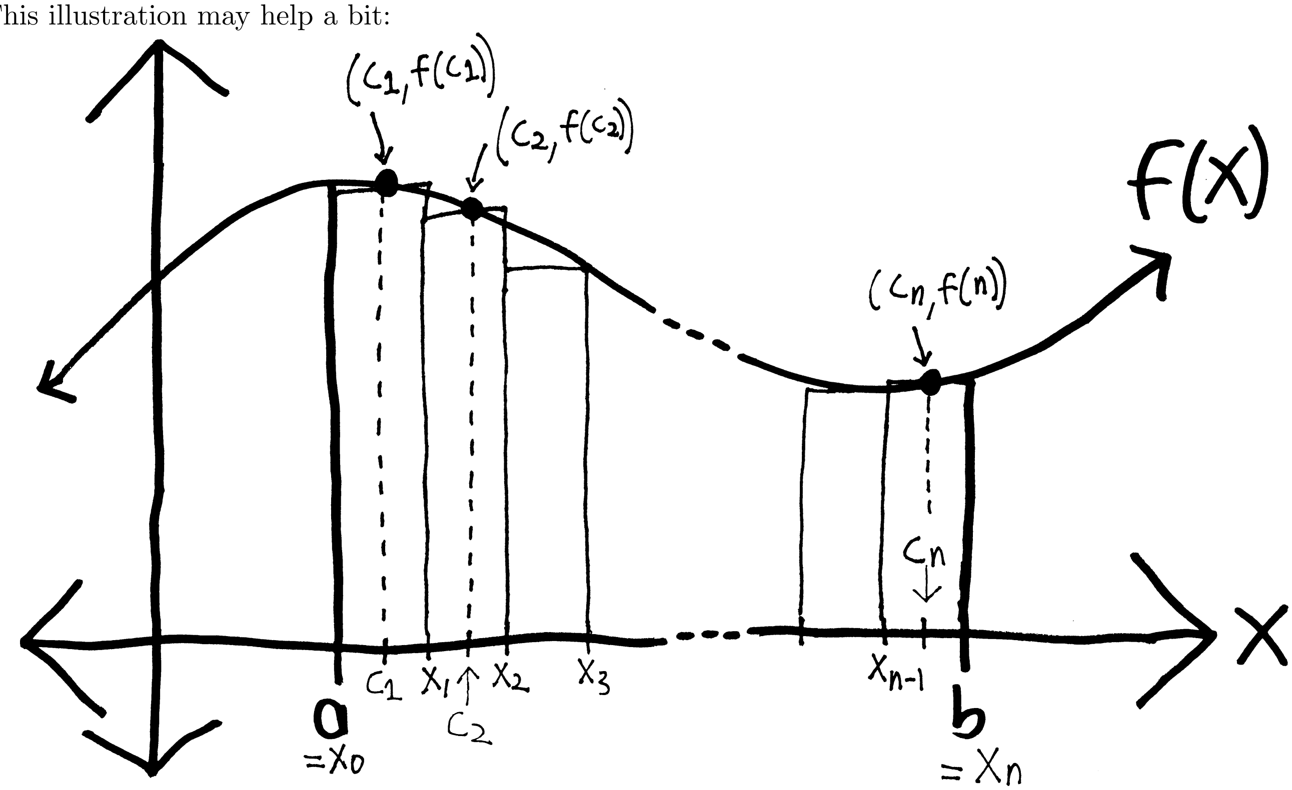
go:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left[\sum_{k=1}^{k=n} f(c_k)\Delta x_k\right]$$

This looks scary, so let's talk about each part bit-by-bit:

- $f(c_k)$ is the height of the kth box
  - and so c_k is, like, the x-coordinate from which we want to measure the height,
- and Δx_k is the width of the kth box,
  - i.e., the difference (hence the Δ) between the x-coordinates of box k and box $k-1$
  - i.e., $x_k - x_{k-1}$
- so, thus, $f(c_k)\Delta x_k$ = (height of the kth box)·(width of the kth box) = area of the kth box
- and we want to add up all n of the boxes, so we take a $\sum$ from the first box to the nth box
- and we take a limit as $n \to \infty$, because we want this to be a Riemann sum not with a finite number of boxes—we don't want to *approximate* the area—we want this to be a Riemann sum with an *infinite* number of boxes—we want the *exact* area!

This illustration may help a bit:



The Fundamental Theorem of Calculus

He computed with great fascination,
The Riemann sum approximation.
But he knew he could get,
The best answer yet,
With definite integration.

what's weird is that we don't directly use the definition we sort of attack this from the side the definition was kind of just for fun

The Fundamental Theorem of Calculus, Part I: $\dfrac{d}{dx}\left[\displaystyle\int_0^x f(t)\,dt\right] = f(x)$

This tells us, more or less, that an integral is an antiderivative, at least when we choose the bounds correctly. It tells us that if we take a derivative of an integral, we get the function that was inside the integral—that $\frac{d}{dx}$ and $\int$ cancel each other out.

(Of course, the caveat is that we have an integral going from some fixed constant a to the variable x. (And note that we've written the function on the inside of the integral as $f(t)$, rather than $f(x)$, to avoid ambiguity between x's.) This should highlight one of the superficial differences between integrals and derivatives: when we find a derivative, we find the derivative at *any* point on a function; our derivative is a function of x. But with an integral, it doesn't really make sense to find the "area" at a single point: we really do need two points between which to find the area.)

Proof: Imagine we have this integral:

$$\int_0^x f(t)\,dt$$

We'll be using this a lot in the proof, so for convenience, let's abbreviate it as $F(x)$:

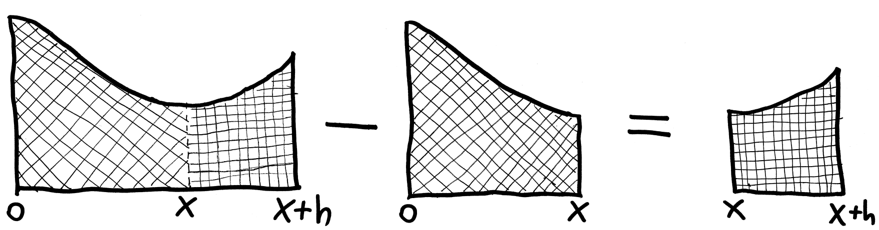
$$F(x) = \int_0^x f(t)\,dt$$

Then: what if we consider:

$$\int_0^{x+h} f(t)\,dt - \int_0^x f(t)\,dt$$

Using one of our geometric properties of integrals, this must be just equal to

$$\int_0^{x+h} f(t)\,dt - \int_0^x f(t)\,dt = \int_x^{x+h} f(t)\,dt$$



Moreover, it must also be equal to $F(x+h) - F(x)$:

$$\underbrace{\int_0^{x+h} f(t)\,dt}_{F(x+h)} \quad - \quad \underbrace{\int_0^x f(t)\,dt}_{F(x)} = F(x+h) - F(x)$$

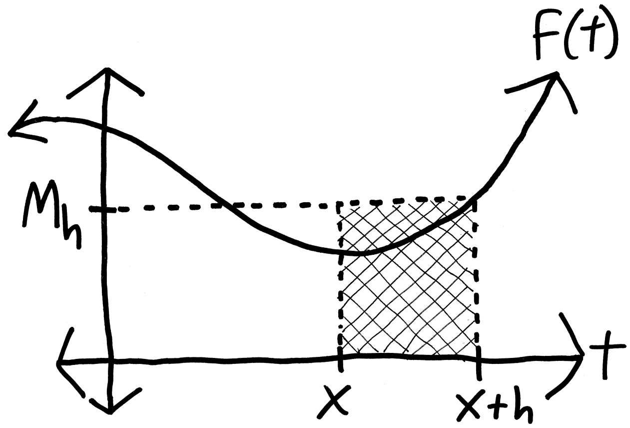
so, combining both of those two ideas:

$$\int_x^{x+h} f(t)\,dt = F(x+h) - F(x)$$

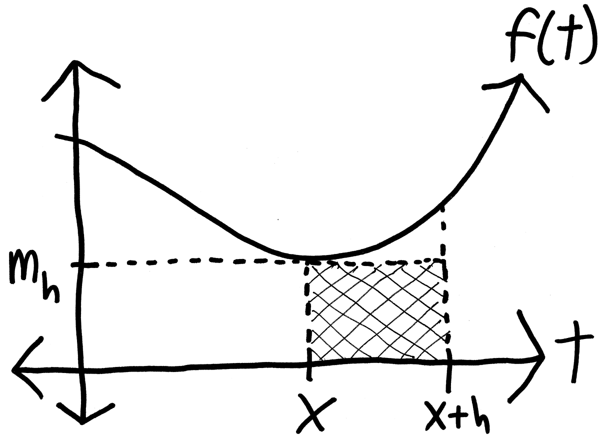
Now: let's make a Riemann sum for this integral! In particular, let's make upper and lower sums! But let's be boring, and make an upper sum that consists only of one box, and a lower sum that also consists only of one box.

- Let's say that M_h is the *maximum* value of f between x and $x+h$

- so then $M_h \cdot h$ is an upper sum for f between x and $x+h$. It's not a very interesting upper sum—it's just a single, giant box encompassing the entire area—but it's an upper sum nonetheless.



- Likewise, let's say that m_h is the *minimum* value of f between x and $x+h$
  - then, by the same reasoning, $m_h \cdot h$ is a lower sum for f between x and $x+h$.



So we have an upper sum and a lower sum, and we know that the actual area beneath the function—the actual integral—must lie between the upper sum and the lower sum:

$$\text{lower sum} \leq \text{actual area} \leq \text{upper sum}$$

or just:

$$m_h \cdot h \leq \int_0^{x+h} f(t)\, dt \leq M_h \cdot h$$

or just:

$$m_h \leq \frac{1}{h} \int_0^{x+h} f(t)\, dt \leq M_h$$

but we know that $\int_0^{x+h} f(t)\, dt$ is just a more complicated way of writing $F(x+h) - F(x)$:

$$m_h \leq \frac{1}{h}\big[F(x+h) - F(x)\big] \leq M_h$$

$$m_h \leq \frac{F(x+h) - F(x)}{h} \leq M_h$$

AND if we make h really small....

$$\lim_{h \to 0} [m_h] \leq \lim_{h \to 0} \left[\frac{F(x+h) - F(x)}{h}\right] \leq \lim_{h \to 0} [M_h]$$

What happens to these three expressions as $h \to 0$? If you think about it geometrically, as h gets close to 0, then the area between x and $x+h$ gets narrower and narrower. And whatever the maximum value of $f(x)$ is between x and $x+h$ gets closer to $f(x)$—there's nowhere else left to go.

Likewise, as $x+h$ approaches x, the minimum value of $f(x)$ gets closer to $f(x)$ itself. If we had no area—if h actually were zero—then there would be only ONE value of the function (between x and $x+0$), and so $f(x)$ would be both the minimum and maximum value.
So we have:

- $\lim_{h\to 0}[M_h] = f(x)$, and
- $\lim_{h\to 0}[m_h] = f(x)$

If we think of this in terms of our inequality, we get:

$$f(x) \leq \lim_{h\to 0}\left[\frac{F(x+h)-F(x)}{h}\right] \leq f(x)$$

But if $\lim_{h\to 0}\left[\frac{F(x+h)-F(x)}{h}\right]$ is both *less than or equal to* $f(x)$, and *greater than or equal to* $f(x)$, then it must be equal to $f(x)$! (This is the same argument (the "two policeman theorem") we made in proving that the derivative of sine is cosine: we have something sandwiched in the middle of an inequality, and we use a limit to crush everything together.)

So we must have

$$f(x) = \lim_{h\to 0}\left[\frac{F(x+h)-F(x)}{h}\right]$$

Or, put differently:

$$f(x) = \frac{d}{dx}[F(x)]$$

Or, yet differently:

$$f(x) = \frac{d}{dx}\left[\int_0^x f(t)\,dt\right]$$

But if the derivative of $\int_a^x f(t)\,dt$ is $f(x)$, then that's the same as saying that an antiderivative of $f(x)$ is $\int_0^x f(t)\,dt$.

The F.T.C., Part II: If $F(x)$ is any antiderivative of $f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$

Proof: From Part I, we already know that if $F(x)$ is an antiderivative of $f(x)$, then

$$F(x) = \int_0^x f(t)\,dt$$

So then we must have:

$$F(b) = \int_0^b f(t)\,dt$$

and

$$F(a) = \int_0^a f(t)\,dt$$

But then, because of this geometric property of integrals that we've discussed so many times before, we must have:

$$\begin{aligned} \underbrace{\int_0^b f(t)\,dt}_{F(b)} - \underbrace{\int_0^a f(t)\,dt}_{F(a)} &= \int_a^b f(t)\,dt \\ &= F(b) - F(a) \end{aligned}$$

Moreover, it doesn't matter which antiderivative we choose. We already know that functions can have more than one antiderivative, and that different antiderivatives of the same function only differ by a constant. Imagine that both $F(x)$ and $G(x)$ are antiderivatives of $f(x)$. Then, since they only differ by a constant, we must have:

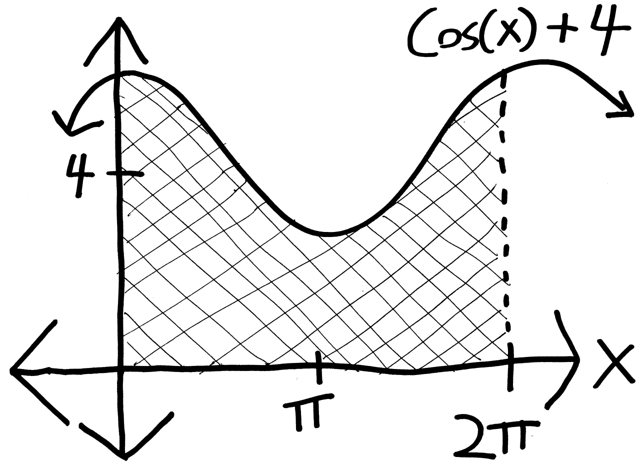
$$F(x) = G(x) + C$$

for some constant C. But then if we consider $F(b) - F(a)$...

$$\begin{aligned} F(b) - F(a) &= [G(b) + C] - [G(a) + C] \\ &= G(b) - G(a) \end{aligned}$$

The constant goes away when we subtract, since plugging in a different value for x (either a or b) doesn't change it. And we get the same thing.

So, for instance, if we wanted to find the area underneath the function $f(x) = \cos(x) + 4$ from 0 to 2π, we'd just use this formula.



The antiderivative of $f(x) = \cos(x) + 4$ is $F(x) = \sin(x) + 4x$, and so if we plug 0 and 2π in, we get

$$\begin{aligned} \int_0^{2\pi} \cos(x)\,dx &= \underbrace{(\sin(2\pi) + 4\cdot 2\pi)}_{F(b)} - \underbrace{(\sin(0\pi) + 4\cdot 0)}_{F(a)} \\ &= (0 + 8\pi) - (0 + 0) \\ &= 8\pi \end{aligned}$$

But if integrals are really just antiderivatives... **I guess that means we're going to need to talk more about antiderivatives**.