Some Comments on Solids of Revolution

Calculus 12, Veritas Prep.

So far, you've learned one formula to find the volume of a solid of revolution:

$$\text{the volume of the shape generated by } f(x) \text{ from } a \text{ to } b \text{ revolved around the } x\text{-axis} = \int_a^b \pi(\,f(x)\,)^2\,dx$$

Explained differently, this formula revolves a shape around whatever axis the function defining that shape is written in terms of. For example, if we plug $y = x^2$ into this formula, it revolves x^2 around the x-axis; if we plug $x = \sqrt[3]{y}$ into this formula (and find $\int \pi(\,\sqrt[3]{y})^2\,dy$), it'll revolve $\sqrt[3]{y}$ around the y-axis.

We can derive another formula that does something similar, but with a subtle and important difference:

$$\text{the volume of the shape generated by } f(x) \text{ from } a \text{ to } b \text{ revolved around the } y\text{-axis} = \int_a^b 2\pi x f(x)\,dx$$

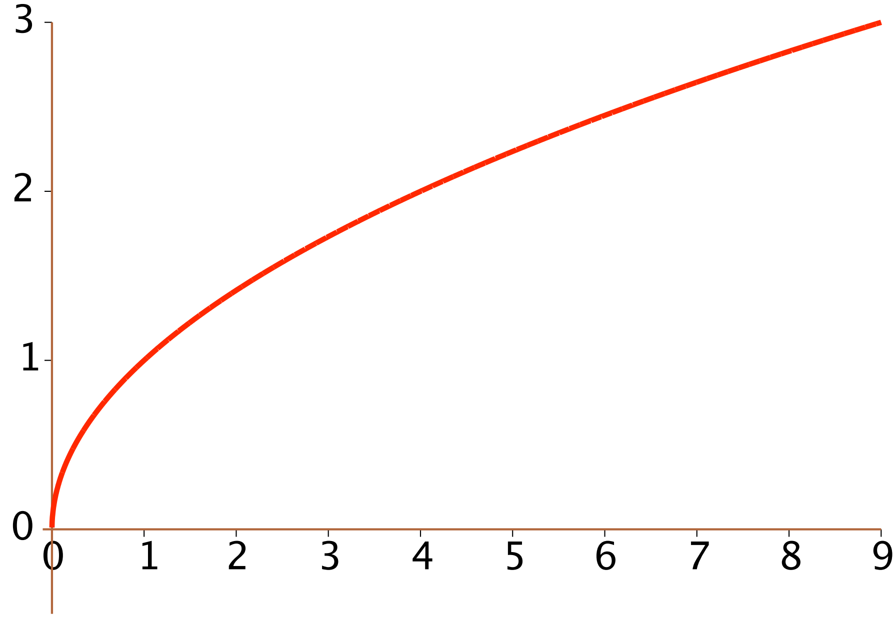
This formula is different. This formula takes a function written in terms of one variable, and revolves it around the *opposite* axis. It takes functions written in terms of x and revolves them around the y-axis; it takes functions in terms of y and revolves them around the x-axis.

If you're wondering why we care about having two different formulas that do more-or-less the same thing—aren't they redundant? couldn't we just solve a function for either variable and then use either formula to find the volume of the same shape?—skip ahead to the second example.

As another comment: neither of these formulas actually *describe* the three-dimensional shapes. They merely find their volumes. If you wanted to describe the shapes in three dimensions, you'd need a third variable (and some understanding of multivariate calculus). For example—and this is just an example, and not something to generalize from—if you want the equation for a parabola rotated around the y-axis (a three-dimensional *paraboloid*, it's $y = x^2 + z^2$, with z being the axis of our third dimension

Example: The Volume Of The Same Shape Found Using Two Different Formulas

Maybe I want to revolve the bullet-like shape given by the function $y = \sqrt{x}$ from 0 to 9 around the x-axis (and find the volume of said bullet). The 2D (unrevolved) shape looks like this:



I can find the volume of the 3D shape in two ways. Using the first formula, I can just things in, and get:

$$\begin{aligned} V &= \int_0^9 \pi(\sqrt{x})^2 dx \\ &= \int_0^9 \pi x \, dx \\ &= \left[\frac{\pi}{2}x^2\right]_0^9 \\ &= \left(\frac{\pi}{2}9^2\right) - \left(\frac{\pi}{2}0^2\right) \\ &= \frac{81\pi}{2} - 0 \\ &= \frac{81\pi}{2} \end{aligned}$$

Or we could use the second formula. First, we have to write this in terms of y, because that way, when I plug it into my other formula, the formula will revolve it around the x-axis (which is what we want):

$$\begin{aligned} y &= \sqrt{x} \\ y^2 &= (\sqrt{x})^2 \\ x &= y^2 \end{aligned}$$

Then I have to find the new starting and ending points of my shape. It starts at $x = 0$, and it ends at $x = 9$. If I plug those into my original function, I find that those two points are

$$\begin{aligned} \sqrt{x} &= y \\ \sqrt{0} &= 0 \\ \sqrt{9} &= 3 \end{aligned}$$

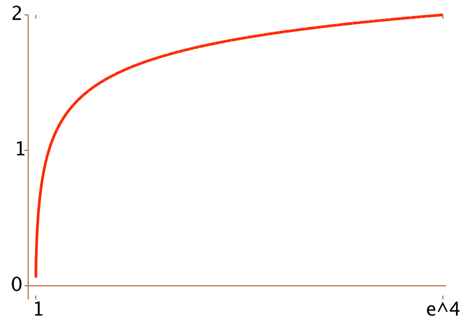
So (with respect to y) the shape goes from $y = 0$ to $y = 3$. So if I plug things into my second formula, I have:

$$\begin{aligned} V &= \int_0^3 2\pi \cdot y \cdot y^2 dy \\ &= \int_0^3 2\pi y^3 dy \\ &= \left[\frac{2\pi}{4}y^4\right]_0^3 \\ &= \left(\frac{2\pi}{4}3^4\right) - \left(\frac{2\pi}{4}0^4\right) \\ &= \frac{81\pi}{2} - 0 \\ &= \frac{81\pi}{2} \end{aligned}$$

Same answer!

Another Important Example

What if we want to take the region bounded by $y = \sqrt{\ln(x)}$ and $y = 0$ (on the top and bottom) and $x = 1$ and $x = e^4$ (on the left and right, revolve it around the x-axis, and find the volume? The region, by the way, looks like this:



So the 3D shape will look like a much-less-aerodynamic bullet. If we use our first formula, we get

$$V = \int_1^{e^4} \pi(\sqrt{\ln(x)})^2 dx = \int_1^{e^4} \pi \ln(x) dx$$

How do you work this out? Who knows? Not you. You have no idea how to do this integral[1]. GOOD THING WE HAVE ANOTHER FORMULA! What if we try using it? First we'd need to put everything in terms of y. So we'll have:

$$y = \sqrt{\ln(x)} \Longrightarrow y^2 = \ln(x) \Longrightarrow e^{(y^2)} = x \Longrightarrow x = e^{(y^2)}$$

$$x = 1 \Longrightarrow y = \sqrt{\ln(1)} \Longrightarrow y = \sqrt{0} \Longrightarrow y = 0$$
$$x = e^4 \Longrightarrow y = \sqrt{\ln(e^4)} \Longrightarrow y = \sqrt{4} \Longrightarrow y = 2$$

Then if we plug this stuff into our other formula, it'll revolve this shape around the x-axis (which is what we want). So we'll have:

$$\begin{aligned} V &= \int_1^2 2\pi \cdot y \cdot e^{y^2}\, dy \\ &= \left[\pi e^{y^2}\right]_1^2 \\ &= (\pi e^{2^2}) - (\pi e^{1^2}) \\ &= \pi e^4 - \pi e \\ &= \pi e(e^3 - 1) \\ &\approx 179.88 \end{aligned}$$

Yay!

[1] I know how to do it! Eventually, you will, too. But not yet.

One Last Thing

Note that we can generalize both of our formulas. What if we want to take a shape generated by rotating the area *between* two functions around the x-axis, and find the volume of that shape? (Like, we could have a ring or something. A "hollow" of revolution, rather than a solid of revolution.) Not surprisingly, we simply subtract two integrals (the volume of the outer shape minus the volume of the inner shape):

$$\begin{array}{c}\text{the volume of the shape generated by rotating}\\ \text{the area between } f(x) \text{ and } g(x) \text{ between } x = a\\ \text{and } x = b \text{ around the } x\text{-axis}\end{array} = \int_a^b \pi(\,f(x)\,)^2\,dx - \int_a^b \pi(\,g(x)\,)^2\,dx$$

Note here that, as usual, the order of the functions is important—$f(x)$ should be the outer radius of this shape, and $g(x)$ the inner radius. Otherwise you'd get a negative you didn't want. And if $g(x) = 0$ (i.e., is the x-axis), then we just have the same formula we had before. We can make the same generalization for our other method:

$$\begin{array}{c}\text{the volume of the shape generated by rotating}\\ \text{the region between } f(x) \text{ and } g(x) \text{ between}\\ x = a \text{ and } x = b \text{ around the } y\text{-axis}\end{array} = \int_a^b 2\pi x\,f(x)\,dx - \int_a^b 2\pi x g(x)\,dx$$