Sequences and Series; Finite and Infinite

Calculus 12, Veritas Prep.

We are about to do the coolest theorem in calculus. Not the most important theorem, mind you—that's the FTC—the coolest. We are about to do the coolest, weirdest, most mind-blowing theorem in all of first-year calculus. I have stories about this theorem that I'd be fired if I told you. We are about to do **Taylor's theorem**. Taylor's theorem, roughly speaking, says that ***everything is a polynomial***. Trig functions, exponential functions, logarithms, rational functions, all sorts of weird stuff—they can all just be thought of as polynomials (albeit possibly infinitely-long polynomials). And that is wonderful and weird, because polynomials are *really nice functions*. They're continuous everywhere—they have no holes or asymptotes—they have derivatives everywhere—we can plug any real number in—they're nice and predictable and curvy. They're so simple! Remember how easy graphing them is, compared to graphing rational functions? All you do is factor them! Or think about how easy it is to evaluate a polynomial at any random point—you just plug in the number, do a lot of arithmetic, and eventually get it! You can't do that with trig functions (unless you're lucky and you want to plug in, say $\pi/2$):

- if $f(x) = x^2 + 3x + 5$, and I want to find $f(2)$, I just plug 2 in for x: $f(2) = 2^2 + 3 \cdot 2 + 5 = 15$
- if $f(x) = \ln(x)$, and I want to find $f(2)$, I just plug 2 in for x: $f(2) = \ln(2) =$ ...um. I don't know how to work this out as a decimal.

Anyway. Taylor's theorem says that everything is a polynomial[1]. For instance, sine is a polynomial—albeit an infinitely-long one:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

So is e^x:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

This stuff is really cool! But before we start talking about it fully, there are some preliminaries we need to take care of. Namely: Taylor series are a type of series, so we should probably talk about those, and series are made of individual terms (a sequence of terms, added together), so we should probably talk about those, too.

Sequences

A sequence, in essence, is just an ordered list of numbers. Here are some examples of sequences:

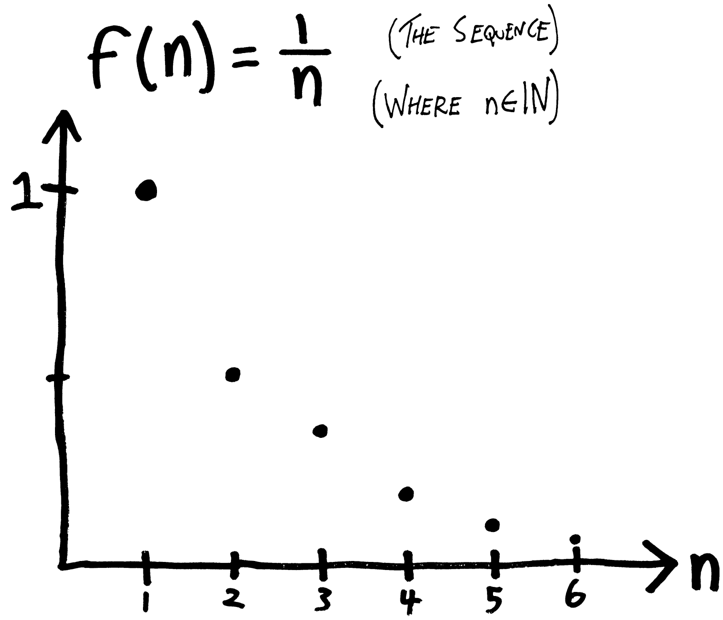
- $1, 2, 3, 4, 5, 6 \cdots$ (the natural numbers)
- $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ (fractions!)
- $-1, 1, -1, 1 \cdots$ (it's flipping back and forth!)
- $3, 3, 3, 3, 3 \cdots$ (kind of boring)
- $1, 4, 9, 16, 36 \cdots$ (squares!)
- $2, 4, 6, 8, \cdots$ (even numbers!)
- $x, x^2, x^3, x^4, \cdots$ (a sequence of functions, rather than numbers!)

[1] If you're a math person, please don't ruin the fun by shouting out some idiotic comment. Look, I know the cops are going to break up this party eventually and cite us for violating uniform convergence, and then in the morning we'll all wake up with deadly hangovers as a result of having approximated, say, $\ln(1 + x)$ far outside its radius of convergence... but for now, let us just believe in the beauty and the glory of EVERYTHING BEING POLYNOMIALS. Because it is beautiful. Tomorrow will come when it will.

Note that these are all examples of **infinite sequences**, i.e., sequences that could go on forever. Question: I've listed, for each sequence, the first term, the second term, the third term, etc., etc., ... what will the nth term look like?

- $1, 2, 3, 4, 5, 6 \cdots, n, \cdots$
- $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{n}, \cdots$
- $-1, 1, -1, 1, -1 \cdots, (-1)^n, \cdots$
- $3, 3, 3, 3, 3 \cdots, 3, \cdots$
- $1, 4, 9, 16, 36 \cdots, n^2, \cdots$
- $2, 4, 6, 8, \cdots, 2n, \cdots$
- $x, x^2, x^3, x^4, \cdots, x^n, \cdots$

Another way of thinking about a sequence—and this is the way we'd formally define it—is that it's a function, but whereas basically all of the functions we deal with can take any real number as input, a sequence can only take a natural number as input. It's a function whose domain is only the natural numbers (but whose range could be all real numbers). Here's a picture of what the sequence $f(n) = 1/n$ looks like:



If I want to fully describe a sequence, then rather than list terms, it's probably a better idea to just give the formula for the nth term:

- $S_n = n$ or $f(n) = n$ is the sequence $1, 2, 3, 4, 5, 6 \cdots, n, \cdots$
- $S_n = \frac{1}{n}$ or $f(n) = \frac{1}{n}$ is the sequence $1, \frac{1}{2}. \frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{n}, \cdots$
- $S_n = (-1)^n$ or $f(n) = 2n$ is the sequence $1, -1, 1, -1, 1 \cdots, (-1)^n, \cdots$
- $S_n = 3$ or $f(n) = 3$ is the sequence $3, 3, 3, 3, 3 \cdots, 3, \cdots$
- $S_n = n^2$ or $f(n) = n^2$ is the sequence $1, 4, 9, 16, 36 \cdots, n^2, \cdots$
- $S_n = 2n$ or $f(n) = 2n$ is the sequence $2, 4, 6, 8, \cdots, 2n, \cdots$
- $S_n = x^n$ or $f(n) = x^n$ is the sequence $x, x^2, x^3, x^4, \cdots, x^n, \cdots$

Note that I'm using the letter n rather than x because, well, mathematicians tend to use the letter n (or sometimes k, or, less often, m, i, and j) to represent natural numbers, whereas x tends to be used to represent real numbers. Of course, that's not a necessary convention; it's just usual practice. The context of the problem should make it clear whether a variable ranges over all possible real numbers (like the x in $f(x) = x^2$) or over just the natural numbers (like the n in $S_n = 2n$). Remember that $\mathbb{N}$ is the symbol for the natural numbers, whereas $\mathbb{R}$ is the symbol for real numbers. By the fancy expression in the above graph "$n \in \mathbb{N}$," I mean that n is a natural number, or, more literally, n is a member of the set of natural numbers, with $\in$ meaning "is an element/member of."

The main question we can ask about a sequence is: does it **converge** or **diverge**? Meaning: as $n \to \infty$, does the sequence either:

a) approach some single, finite number (**converge**), or

b) not? (**diverge**)

And as a secondary question: if it does converge, **what does it converge to**?

For example:

- The sequence $S_n = n$, i.e., the sequence $1, 2, 3, 4, \cdots$ diverges, because as $n \to \infty$, the sequence just keeps getting bigger and bigger, i.e., $S_n \to \infty$.
- The sequence $S_n = 1/n$ (i.e., $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$) converges, because as $n \to \infty$, $S_n \to 0$. So it converges, and moreover, it converges to 0.
- The sequence $S_n = \frac{(-1)^n}{n}$, i.e., the sequence $-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \frac{-1}{5}, \cdots$ converges, because as $n \to \infty$, $S_n \to 0$, just as in the last example. Note that, because of the $(-1)^n$, it bounces around—it keeps alternating between being positive and negative. But it still approaches 0.
- The sequence $S_n = (-1)^n$, i.e., the sequence $-1, 1, -1, 1, -1, \cdots$ diverges, because as $n \to \infty$, the limit of S_n doesn't exist. It keeps alternating between $+1$ and -1. It never settles down.
- The sequence of functions $S_n = x^{1/n}$, i.e., the sequence $1, x, x^{1/2}, x^{1/3}, x^{1/4}, \cdots$ converges to x^0, or just 1.

Series

So, those are sequences. They're kind of cool, but not that cool, because they're easy to deal with. If we want to find whether a sequence converges, we just take a limit! Not too difficult. But what if we add all the terms in a sequence together? For instance:

$$\text{rather than having the sequence: } 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$$

$$\text{what if we have the series: } 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

This is a very different creature. We know that the sequence converges, but we don't know whether the series (in which we add everything together) converges. More formally, a **series**, or a **sum**, is when we add the terms of a sequence together, rather than simply listing them. They could be finite (I add together a finite number of terms) or infinite (I add together an infinite number of terms). Here's an example of a finite series: $5 + 10 + 15 + 20$

There are three different ways we can write series:

$$\begin{aligned} \text{I could show each term explicitly:} \quad & 5 + 10 + 15 + 20 \\ \text{I could write it as a single number:} \quad & = 50 \\ \text{or I could write it using } \Sigma\text{-notation for sums:} \quad & = \sum_{k=1}^{k=4} 5k \end{aligned}$$

You've seen $\sum$-notation before, but as a quick reminder, it works something like this:

$$\sum_{k=1}^{k=4} 5k = \underbrace{5\cdot 1}_{\text{the } k=1 \text{ term}} + \overbrace{5\cdot 2}^{\text{the } k=2 \text{ term}} + \underbrace{5\cdot 3}_{\text{the } k=3 \text{ term}} + \overbrace{5\cdot 4}^{\text{the } k=4 \text{ term}} = 5 + 10 + 15 + 20 = 50$$

Or, more generally:

$$\sum_{k=a}^{k=b} f(k) \;=\; f(a) + f(a+1) + f(a+2) + \cdots + f(b-2) + f(b-1) + f(b)$$

Obviously, if I have an infinite series, I can't write every term explicitly—there are an infinite number of them! But I can ask the same question I ask of sequences: given a series, **does it converge or diverge?** For series, this is a much harder question to answer[2]. With sequences, we can just take a limit; with series, we have just a smattering of *ad hoc* methods, some of which might or might not work for a given series.

There are some obvious considerations first. Consider the infinite series:

$$\sum_{n=1}^{n=\infty} n = 1 + 2 + 3 + 4 + 5 + 6 + \cdots$$

Obviously this diverges. We're adding bigger and bigger numbers! The sum is getting bigger and bigger! So clearly, if each of the individual terms in the series increase, the series will diverge. In fact, even if each of the terms stay the same, the series diverges:

$$\sum_{n=1}^{n=\infty} 5 = 5 + 5 + 5 + 5 + 5 + \cdots = \infty$$

So we can say this: **if the individual terms of a series do not decrease (i.e., increase or remain constant), the sequence will diverge**. But just because the terms of a series decrease, it is not always the case that the series converges. For example, consider the two series:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

The terms in both of these series go to 0. But (for reasons which you're about to see) one of these series converges, and the other diverges.

One last thing: don't get the terms "sequence" and "series" confused. I do it all the time. Don't follow my example:

This is a sequence: $1, 2, 3, 4, 5, 6, \cdots, n, \cdots$

This is a series: $1 + 2 + 3 + 4 + 5 + 6 + \cdots + n + \cdots$

A Cool, Simple Series

Consider the series:

$$\sum_{n=0}^{n=\infty} a^n = 1 + a + a^2 + a^3 + a^4 + \cdots$$

for some number a; for instance, if $a = 1/2$, this becomes the series:

$$\sum_{n=0}^{n=\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

[2] Also note that it's really only an interesting question to ask about *infinite* series, since finite series obviously converge to whatever they add up to.

There is a startling result about this series: *it converges to* $\dfrac{1}{1-a}$:

$$\sum_{n=0}^{n=\infty} a^n = 1 + a + a^2 + a^3 + a^4 + \cdots = \frac{1}{1-a}$$

So, for instance, in the case of $a = 1/2$, no matter how many terms we add, the series simply gets closer and closer to 2:

$$\sum_{n=0}^{n=\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \frac{1}{1-1/2} = \frac{1}{1/2} = 2$$

Let's prove this. Imagine, for the sake of convenience, we call this series S:

$$S = 1 + a + a^2 + a^3 + a^4 + \cdots$$

Then: what happens if we multiply both sides of this equation by a? We'll have:

$$aS = a + a^2 + a^3 + a^4 + a^5 \cdots$$

What if we consider the quantity $S - aS$? This must be just:

$$S - aS = (\overbrace{1 + a + a^2 + a^3 + a^4 + \cdots}^{S}) - (\overbrace{a + a^2 + a^3 + a^4 + a^5 \cdots}^{aS})$$

But if I combine those two parentheses, most nearly everything will cancel out—we've got an a, and a $-a$; we've got an a^2, and a $-a^2$, etc. The only thing that will be left is 1:

$$S - aS = 1$$

But we can easily solve this for S:

$$S(1-a) = 1$$

$$S = \frac{1}{1-a}$$

Ta-da! Now we can deal with any series of the form $\sum a^n$. In fact, we have quite a bit of power, because most of our tests for convergence only tell us whether a certain series converges—they don't tell us what such series converge *to*. But this test does. For instance, if we have the series:

$$1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \cdots$$

which is really just the series:

$$\sum_{n=0}^{n=\infty} \left(\frac{2}{3}\right)^n$$

we know, using this formula we just proved, that this series must converge, and it must converge to 3:

$$\sum_{n=0}^{n=\infty} \left(\frac{2}{3}\right)^n = \frac{1}{1-2/3} = \frac{1}{1/3} = 3$$

Unfortunately (or fortunately, depending on how you look at it), there's a pretty substantial caveat. Namely: what if we have the series $\sum 2^n$:

$$\sum_{n=0}^{n=\infty} (2)^n = 1 + 2 + 4 + 8 + 16 + \cdots$$

Does this series converge? Certainly not! It just keeps getting bigger and bigger! It goes to ∞! But according to our theorem, it does converge:

$$\sum_{n=0}^{n=\infty} (2)^n = \frac{1}{1-2} = -1$$

This series is not -1. You cannot keep adding powers of two and get -1. That is utterly ridiculous. You will keep getting bigger and bigger numbers. The problem is pervasive—consider $a = 1$:

$$\sum_{n=0}^{n=\infty} (1)^n = 1 + 1 + 1 + 1 + 1 + \cdots = \infty$$

So really, we should specify that in order for this series to converge, a must be less than 1. And, actually, we should say that a has to be between -1 and 1, because numbers below -1 give us the same problem:

$$\sum_{n=0}^{n=\infty} (-1)^n = 1 - 1 + 1 - 1 - 1 - \cdots = \text{doesn't converge! keeps flipping between 0 and 1}$$

$$\sum_{n=0}^{n=\infty} (-2)^n = 1 - 2 + 4 - 8 + 16 - \cdots = \text{doesn't converge! gets bigger and bigger, but with alternating signs}$$

So we should restate our theorem a little bit more precisely:

$$\sum_{n=0}^{n=\infty} a^n = \frac{1}{1-a}, \text{ but only if } -1 < a < 1$$

This type of series, by the way, is known as a **geometric series**. Not that the name really means much, but other people use the name, so you should probably know it.

Another Type of Series

Here's another series: what if I have something like:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

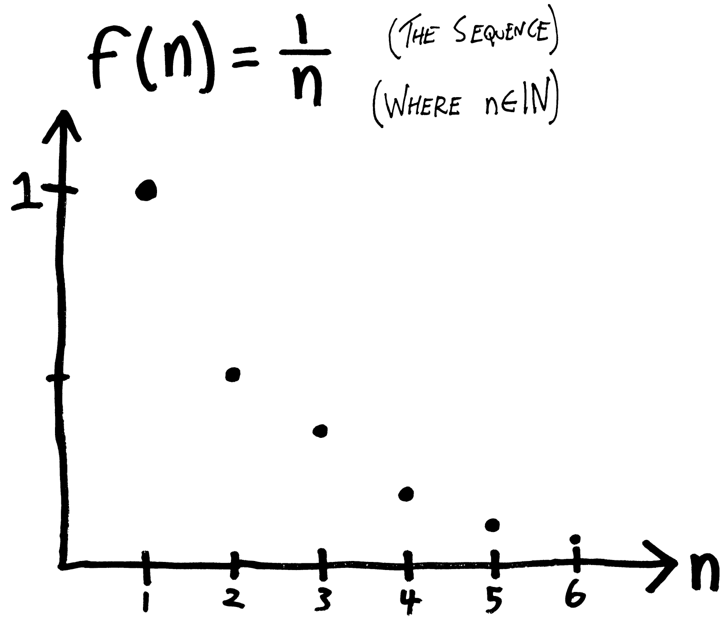
or, written in $\sum$-notation:

$$\sum_{n=1}^{n=\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

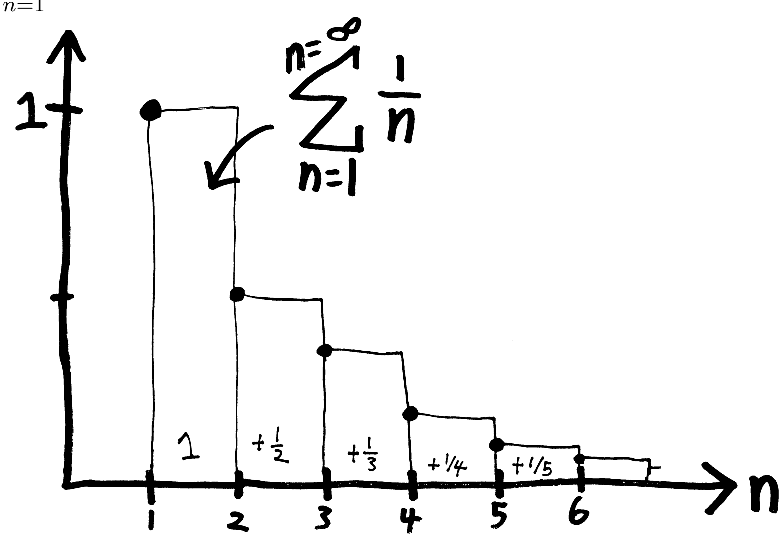
(This is called the **harmonic series**, by the way.) Does this converge? It might, since the terms get smaller. It doesn't obviously diverge. But how do we deal with it? Allow me to suggest this: did the last series remind you at all of an integral? an improper integral? remember how weird it was that sometimes we have integrals to ∞ that converge? i.e., shapes that are infinitely long but have finite area? like, for example:

$$\int_1^\infty \frac{1}{x^2} = 1$$

And if we have a series, it's kind of like adding up areas! Think about it like this: remember that the sequence $f(n) = 1/n$ looks like this (just a bunch of points):

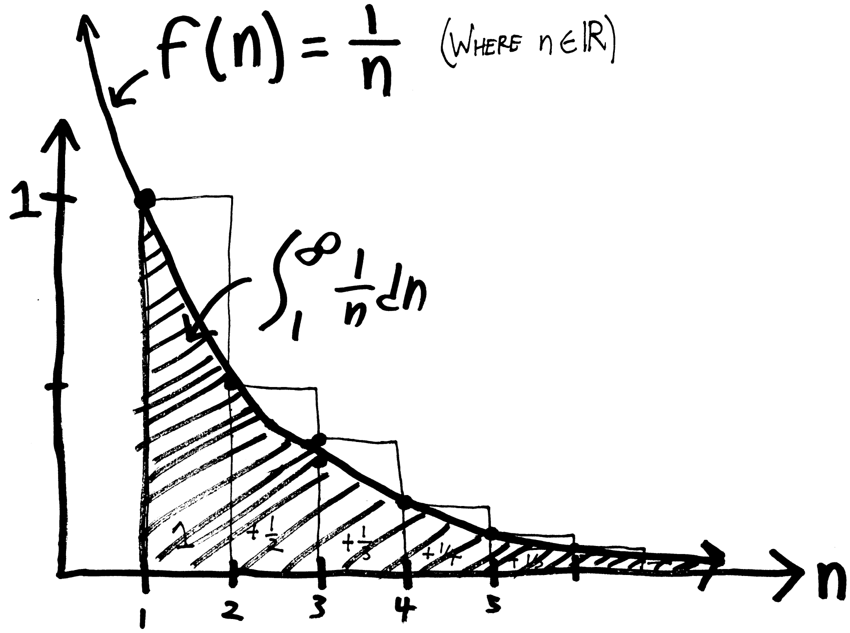


but if we turn each of these points into the top of rectangles, and then find the areas of all the rectangles, that's like the series $\sum_{n=1}^{n\infty} \frac{1}{n}$:



So I guess we could make the analogy: a series is to a sequence as an integral is to a smooth curve.

Anyway, what if we simultaneously consider the function $f(n) = 1/n$, where n can be *any real number* (not just a natural number), and then find the area underneath that curve?



Clearly, there will be slightly less area underneath the curve $1/n$ than there is in the boxes (i.e., in the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$). Put differently, we know that:

$$\int_1^{\infty} \frac{1}{n}\, dn < \sum_{n=1}^{n=\infty} \frac{1}{n}$$

But we know how to work out that integral! We know:

$$\int_1^{\infty} \frac{1}{n}\, dn = [\ln(n)]_1^{\infty} = \ln(\infty) - \ln(1) = \infty$$

So the area underneath the curve $1/n$ is infinite... but the area underneath the boxes is greater than the area under the curve, meaning the boxes must have infinite area, too, and since the boxes *are* the series, the series must be infinite, too, and thus the series diverges!

$$\int_1^{\infty} \frac{1}{n}\, dn < \sum_{n=1}^{n=\infty} \frac{1}{n}$$
$$\infty < \sum_{n=1}^{n=\infty} \frac{1}{n}$$
$$\text{so} \sum_{n=1}^{n=\infty} \frac{1}{n} = \infty$$

In fact, we can state this as a **more general principle**:

$$\text{if} \int_a^{\infty} f(n)\, dn \text{ diverges, then } \sum_{n=a}^{n=\infty} f(n) \text{ also diverges}$$
$$\text{if} \int_a^{\infty} f(n)\, dn \text{ converges, then } \sum_{n=a}^{n=\infty} f(n) \text{ also converges}$$

We could have use this test to figure out that the geometric series (our previous example) converges: we had the series $\sum_0^\infty a^n$, where $0 < a < 1$, and we know:

$$\begin{aligned}\int_0^\infty a^n\,dn &= \left[\frac{1}{\ln(a)}a^n\right]_0^\infty \\ &= \frac{1}{\ln(a)}a^\infty - \frac{1}{\ln(a)}a^0 \\ &= \frac{1}{\ln(a)}\cdot 0 - \frac{1}{\ln(a)}\cdot 1 \\ &= \frac{-1}{\ln(a)}, \text{ which is finite}\end{aligned}$$

so since $\int_0^\infty a^n\,dn$ converges, $\sum_0^\infty a^n$ must converge, too. The disadvantage is that using this test, we know that it converges, but we don't know what it converges to. It *doesn't* converge to $\frac{-1}{\ln(a)}$, because a) we already know it converges to $1/(1-a)$, and clearly those two things are not equal, and b) this is because an integral is not precisely the same as a sum—there's extra area in the boxes that the smooth curve of the integral misses (look at the previous graph—the area of the integral and of the sum are similar, but not identical).

Plus, this test won't always work. What if we have a series like $\sum_{n=0}^{\infty}\frac{1}{n!}$? We can't consider $\int_0^\infty \frac{1}{n!}\,dn$, because we have no idea how to antidifferentiate $n!$. Likewise with, say, $\sum_{n=0}^{\infty}\frac{1}{(-1)^n}$, since we can't antidifferentiate that, either. (Our rule for $\int a^x dx$ only works when $a > 0$, because otherwise the function is really really bizarre and somewhat intractable.)

Another Test For Convergence/Divergence

Here's another cool series:

$$1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \cdots$$

Any idea what its formula is? No? It's a factorial! It's $1/n!$:

$$\sum_{n=1}^{n=\infty}\frac{1}{n!} = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \cdots$$

So the obvious question we can ask about this series, like about every other series, is: does it converge? Each of the terms are getting smaller, so it might. But we can't come up with a nice little algebraic formula for its sum, like we could for a geometric series. We can't treat it like an integral and see whether the integral converges, because we don't know how to antidifferentiate $n!$—and it's not just that we don't know how to, either, but that the very idea of antidifferentiating (or differentiating) such a function makes no sense, because $n!$ isn't a smooth curve like x^2 or e^x. (What is 3.5! ? Is it $3.5 \cdot 2.5 \cdot 1.5 \cdot 0.5$? Is it $3 \cdot 2 \cdot 1$? Does the question not even make sense?)

What we'll have to do is compare it to a series that we *do* know converges. In the last example, we made this argument:

1. we know that the integral $\int_1^\infty \frac{1}{n}\,dn$ diverges;
2. we know that the series $\sum_{n=1}^{n=\infty}\frac{1}{n}$ is greater than $\int_1^\infty \frac{1}{n}\,dn$;
3. therefore, $\sum_{n=1}^{n=\infty}\frac{1}{n}$ diverges.

We can make a similar argument with $\sum 1/n!$. Here's what we know: we know that $n!$ is a skyrocketing function—it goes up really, really fast. Compare it, for example, with x^2 (or n^2) and 2^n:

n	n^2	2^n	$n!$
1	1	2	1
2	4	4	2
3	9	8	6
4	4	16	24
5	25	32	120
6	36	64	720
7	49	128	5,040
$\vdots$	$\vdots$	$\vdots$	$\vdots$

It takes $n!$ slightly longer to get going—it's not in first place until $n = 4$—but after that, it *zooms* up. Which means, of course, that $1/n!$ is zooming down, zooming down to 0 far faster than $1/n^2$ or $1/2^n$. BUT WAIT. We know that $\sum 1/2^n$ converges. That's just a geometric series:

$$\sum_{n=0}^{n=\infty} \frac{1}{2^n} = \sum_{n=0}^{n=\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-1/2} = \frac{1}{1/2} = 2$$

AND we know that

$$\frac{1}{2^n} > \frac{1}{n!}, \text{ at least after } n = 4$$

so then we know that

$$\sum_{n=4}^{n=\infty} \frac{1}{2^n} > \sum_{n=4}^{n=\infty} \frac{1}{n!}$$

(Note how I changed the n's to start at 4. Not a big deal, though. I'll address it more in a moment.) Since the $1/2^n$ series converges, and since $1/n!$ is less than $1/2^n$, the $1/n!$ series must converge, too:

$$\underbrace{\sum_{n=4}^{n=\infty} \frac{1}{2^n}}_{\text{this converges}} \quad > \quad \underbrace{\sum_{n=4}^{n=\infty} \frac{1}{n!}}_{\text{so this must, too}}$$

Of course, our original question was not about $\sum 1/n!$ when n starts at 4; our question was about $\sum 1/n!$ when n starts at 1. But this isn't a concern. If I just add the first few terms back in, I'll just be adding some finite number, and that won't all-of-sudden make the series diverge. Put more formally:

$$\sum_{n=0}^{n=\infty} \frac{1}{n!} = \overbrace{1}^{\text{the } n=1 \text{ term}} + \underbrace{\frac{1}{2}}_{\text{the } n=2 \text{ term}} + \overbrace{\frac{1}{6}}^{\text{the } n=3 \text{ term}} + \sum_{n=4}^{n=\infty} \frac{1}{n!}$$

$$\sum_{n=0}^{n=\infty} \frac{1}{n!} = \frac{10}{6} + \sum_{n=4}^{n=\infty} \frac{1}{n!}$$

We know that the series from $n = 4$ converges, so adding 10/6 won't make it diverge. It will still converge, even if we add 10/6. The sum from $n = 1$ will converge to a slightly different point, yes (a point 10/6 greater than what the sum from $n = 4$ converges to), but since we don't know what the sum from $n = 4$ converges to in the first place, this doesn't really change our situation. The point is:

$$\sum_{n=1}^{n=\infty} \frac{1}{n!} \text{ converges.}$$

Anyway, the point is, this is another method we can use to determine whether series converge or diverge: we can compare them to a series we know something about. We might have a series that's always greater than a series which diverges; therefore, we know it diverges. We might have a series that's always smaller than a series which converges; therefore, we know it converges.

(If we have a series that's always greater than a series which converges; we know nothing, since the series could either converge or diverge (the latter if it's too much greater than the series that converges). If we have a series that's always less than a series that diverges, we know nothing, since the series could either converge or diverge (the latter if it's not smaller enough than the series which diverges).)

Here is a simpler form of the two arguments we could make:

ARGUMENT, VERY BRIEFLY:

We have a series that's always smaller than a series which converges; therefore, we know it converges.

ARGUMENT, SLIGHTLY MORE DETAILED:

1. we know that series B converges;
2. the terms of series B are always greater than the terms of series A;
3. therefore series B is greater than series A;
4. put differently, series A is less than series B;
5. therefore, since series B converges, series A also converges.

ARGUMENT, VERY BRIEFLY:

We have a series that's always greater than a series which diverges; therefore, we know it diverges.

ARGUMENT, SLIGHTLY MORE DETAILED:

1. we know that series B diverges to ∞;
2. the terms of series B are always less than the terms of series A;
3. therefore series B is less than series A;
4. put differently, series A is greater than series B;
5. therefore, since series B diverges, series A also diverges.

We could use this sort of argument for all sorts of different series—for example, we could have a series that we couldn't take an integral of, because we don't know how to antidifferentiate it, but we could compare it to one we did know how to antidifferentiate.

Problems

Find a formula for the nth term of the sequence:

1. $1, -1, 1, -1, 1\cdots$

2. $-1, 1, -1, 1, -1, \cdots$

3. $1, -4, 9, -16, 25\cdots$

4. $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \cdots$

5. $0, 3, 8, 15, 24, \cdots$

6. $-3, -1, -1, 0, 1, \cdots$

7. $1, 5, 9, 13, 17\cdots$

8. $2, 6, 10, 14, 18, \cdots$

9. $2, 3, 5, 7, 11, 13, 17, 19, 23, 29\cdots$

For each of the following sequences, write out the first few terms, and then determine whether it converges or diverges. If it converges, what does it converge to?

10. $S_n = \dfrac{1-n}{n^2}$

11. $f(n) = \dfrac{1}{n!}$

12. $f(n) = \dfrac{(-1)^{n+1}}{2n-1}$

13. $S_n = 2 + (-1)^n$

14. $S_n = 2 + (0.1)^n$

15. $S_n = (-1)^n$

16. $f(n) = \dfrac{1}{(-1)^n}$

17. $S_n = \dfrac{n + (-1)^n}{n}$

18. $S_n = \dfrac{1-2n}{1+2n}$

19. $f(n) = \dfrac{2n+1}{1-3\sqrt{n}}$

20. $S_n = \dfrac{1-5n^4}{n^4+8n^3}$

21. $f(n) = \dfrac{n+3}{n^2+5n+6}$

22. $f(n) = \dfrac{n^2-2n+1}{n-1}$

23. $S_n = \dfrac{1-n^3}{70-4n^2}$

24. $S_n = 1 + (-1)^n$

25. $f(n) = (-1)^n \left(1 - \dfrac{1}{n}\right)$

26. $S_n = \left(\dfrac{n+1}{2n}\right)\left(1 - \dfrac{1}{n}\right)$

27. $S_n = \left(2 - \dfrac{1}{2^n}\right)\left(3 + \dfrac{1}{2^n}\right)$

28. $f(n) = \dfrac{(-1)^{n+1}}{2n-1}$

29. $S_n = \left(-\dfrac{1}{2}\right)^n$

30. $f(n) = \dfrac{\sin(n)}{n}$

31. $f(n) = \dfrac{\sin^2(n)}{2^n}$

32. $S_n = \sqrt{\dfrac{2n}{n+1}}$

33. $f(n) = \sin\left(\dfrac{\pi}{2} + \dfrac{1}{n}\right)$

34. $f(n) = \ln(n) - \ln(n+1)$

35. $f(n) = \dfrac{n}{2^n}$

36. $f(n) = \dfrac{3^n}{n^3}$

37. $f(n) = \dfrac{\ln(n+1)}{\sqrt{n}}$

38. $f(n) = \dfrac{\ln(n)}{\ln(2n)}$

39. $f(n) = 8^{1/n}$

40. $f(n) = (0.03)^{1/n}$

Write out each of the following sums (given in Σ-notation):

41. $\sum_{k=0}^{k=2}(3k+1)$

42. $\sum_{k=0}^{k=3}(2^k)$

43. $\sum_{k=3}^{k=5}\left(\frac{(-1)^k}{k!}\right)$

44. $\sum_{k=2}^{k=4}\frac{1}{3^{k-1}}$

Write each of the following sums in Σ-notation.

45. $1+3+5+7+\cdots+21$

46. $1-3+5-7+\cdots-19$

47. $1\cdot 2+2\cdot 3+3\cdot 4+\cdots+35\cdot 36$

48. $f(x_1)\Delta x_1+f(x_2)\Delta x_2+\cdots+f(x_n)\Delta x_n$

Which of the following series converge, and which diverge? Why? For those that converge, can you determine what they converge to?

49. $\sum_{n=0}^{n=\infty}\left(\frac{1}{5}\right)^n$

50. $\sum_{n=0}^{n=\infty}\frac{1}{5^n}$

51. $\sum_{n=0}^{n=\infty}\frac{1}{2^{n+3}}$

52. $\sum_{n=0}^{n=\infty}\frac{2^{n+3}}{3^n}$

53. $\sum_{n=0}^{n=\infty}\frac{3^{n-1}}{4^{3n+1}}$

54. $\sum_{n=0}^{n=\infty}(-1)^n\left(\frac{1}{5}\right)^n$

55. $\sum_{n=0}^{n=\infty}\left(\frac{1}{\sqrt{2}}\right)^n$

56. $\sum_{n=1}^{n=\infty}\ln(n)$

57. $\sum_{n=1}^{n=\infty}\ln(1/n)$

58. $\sum_{n=1}^{n=\infty}(-1)^{n+1}\frac{3}{2^n}$

59. $\sum_{n=1}^{n=\infty}\left(\sqrt{2}\right)^n$

60. $\sum_{n=0}^{n=\infty}\cos(n\pi)$

61. $\sum_{n=0}^{n=\infty}\frac{\cos(n\pi)}{5^n}$

62. $\sum_{n=0}^{n=\infty}e^{-n}$

63. $\sum_{n=0}^{n=\infty}e^{-2n}$

64. $\sum_{k=0}^{k=\infty}\left(\frac{5}{2}\right)^{-k}$

65. $\sum_{k=0}^{k=\infty}\frac{1}{k^2}$

66. $\sum_{k=0}^{k=\infty}\frac{2}{k^4}$

67. $\sum_{k=0}^{k=\infty}\frac{\ln k}{k^2}$

68. $\sum_{n=1}^{n=\infty}\frac{n^2+1}{n}$

69. $\sum_{n=1}^{n=\infty}(-1)^{n+1}n$

70. $\sum_{n=1}^{n=\infty}\frac{2}{10^n}$

71. $\sum_{n=0}^{n=\infty}\frac{2^n-1}{3^n}$

72. $\sum_{n=1}^{n=\infty}\frac{n!}{1000^n}$

73. $\sum_{n=0}^{n=\infty}x^n$

74. $\sum_{n=0}^{n=\infty}(-1)^n x^n$

75. $\sum_{n=0}^{n=\infty}(-1)^n x^{2n}$

76. $\sum_{k=0}^{k=\infty}\frac{\ln(\sqrt{k})}{k^2}$

77. $\sum_{k=0}^{k=\infty}\frac{1}{2+3^{-k}}$

78. $\sum_{k=0}^{k=\infty}\frac{1}{2+3^k}$

79. $\sum_{n=1}^{n=\infty}\ln\left(\frac{n}{n+1}\right)$

80. $\sum_{n=1}^{n=\infty} \left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} \right)$

81. $\sum_{n=1}^{n=\infty} \left(\frac{1}{2^{1/n}} - \frac{1}{2^{1/(n+1)}} \right)$

82. $\sum_{n=1}^{n=\infty} \left(\frac{1}{\ln(n+2)} - \frac{1}{\ln(n+1)} \right)$

83. $\sum_{k=0}^{k=\infty} \frac{k}{k^2+1}$

84. $\sum_{k=0}^{k=\infty} \frac{1}{3k+2}$

85. $\sum_{n=0}^{n=\infty} e^n$

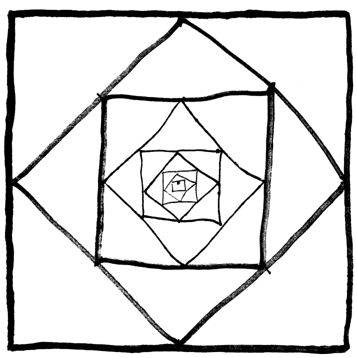
86. $\sum_{k=0}^{k=\infty} ke^{-k^2}$

87. $\sum_{k=0}^{k=\infty} k^2 2^{-k^3}$

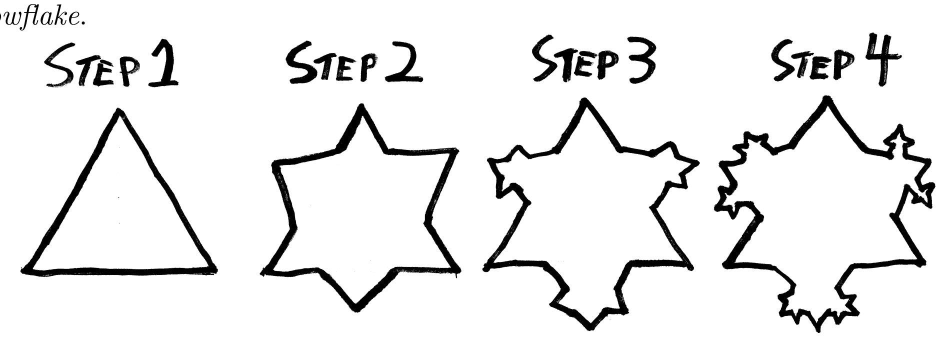
88. $\sum_{n=0}^{n=\infty} \left(\frac{1}{\sqrt{2}} \right)^n$

89. For what values of p does the series $\sum_{k=0}^{k=\infty} k^p$ converge? for what values of p does it diverge?

90. The following is a sketch of a fractal, constructed by taking a square, drawing a square inside of it using the midpoints as corners, drawing a square inside of that using the midpoints as corners, and so forth, *ad infinitum*. Find a) the total area enclosed by all the squares, and b) the total perimeter of all the squares.



91. This is a famous fractal: Helge von Koch's snowflake curve, known generally as the *Koch curve* or *Koch snowflake*.



It is drawn by taking an equilateral triangle, then pasting on three equilateral triangles in the center-third of each side, then pasting on equilateral triangles into the center-third of the sides of the triangles constructed in the previous step, and so forth. Show that a) the total area enclosed by the Koch curve is finite (what is it?), but that b) its perimeter is infinite. (Sorry the sketch isn't better; go online if you want a clearer picture (and animations!).)