In Which We Conclude Calculus by Using Taylor Series to Prove Euler's Identity

Calculus 12, Veritas Prep.

23 February 2011 Name: ______________________________

Directions: Feel free to use scrap paper if you need it. Show all your work neatly and in order, justify each of your steps (use English to explain what you're doing, in addition to symbols), and circle or box your final answers (if applicable). This test totals twenty-seven points. You have two hours. Have fun!

1. (*5 points*) State Taylor's Theorem (i.e., write the Taylor series expansion of some generic function $f(x)$), and explain (informally, in English) what it means and why it is cool.

2. (*4 points*) Being sure to show all your work and not just give the answer, write $\sin(x)$ as a Taylor series. (You could, of course, use any x-value for a, but for convenience, expand it around $x = 0$. Please write it in expanded form (i.e., explicitly as the first few terms) and in Σ-form.)

3. (*4 points*) Write $\cos(x)$ as a Taylor series. (Again, please write it both explicitly and in Σ-form, and show your work.)

4. (*4 points*) Write e^x as a Taylor series (explicitly and in Σ-form, show work, etc.!).

5. (*2 points*) Hey! You know what? We haven't talked about complex numbers at all in this class. It's weird—you learn about them vaguely at some point during high school, but then they go away, and are never seen or heard from again. Well, as it turns out, complex numbers are *awesome*.

I'm going to ramble for a bit—okay? They get called "imaginary numbers" a lot, but that's a horrible name for them, because there's nothing "imaginary" about them. They are every bit as real as 3 or five million or π or $\sqrt{558}$. Which is to say, not at all. If you're a mathematician, numbers are not tangible, physical objects—they exist in some sort of Platonic parallel universe, in which they are known completely by how they are logically defined[1]. To a mathematician, "two" isn't the number of wheels on my bike, or the number of siblings I have—it has a very precise, carefully-constructed definition. Likewise with i. To a mathematician, i is simply $\sqrt{-1}$, and however horrifyingly oxymoronic this may seem, we can still add and subtract and multiple and divide and exponentiate and do all the other things we can do with other numbers—we can still treat $\sqrt{-1}$ as a number without breaking the rest of math.

Now, with most numbers, we can come up with a convenient, physical, real-world metaphor for understanding them. *I have 2 siblings* (a natural number). *There are* -975 *dollars in my bank account* (an integer). *I have eaten* $3/4$ *of your pizza* (a rational number). *The ratio of the circumference of that circle to its diameter is* π (a real number). But we cannot always draw parallels—for example, we tend to think of negatives as somehow representing a deficit or a debt, but what does it mean to say that I've read -5 books? What does it mean for a family to have 2.58 children (viz. *The Phantom Tollbooth*)? At least with numbers like integers and rationals, we can sometimes come up with good physical interpretations. With $\sqrt{-1}$, we can't ever come up with a good physical interpretation[2]. But it still *works* in this abstract, Platonic sense.

On the other hand, if you allow i into your world, things start behaving a bit oddly. For example, consider factoring. If you want to factor a polynomial (into a product of smaller polynomials), there's only one way to do it. For example,

$$x^2 + 9x + 14 = (x + 7)(x + 2)$$

If you want to factor a number (into a product of prime numbers), there's only one way to do it. For example,

$$75 = 3 \cdot 25 = 3 \cdot 5^2$$

But... if you want to factor a number into a product of prime numbers, *and you believe in complex numbers*, you can factor a number *in more than one way*. For example, you could write the number 10 as $5 \cdot 2$, but you can also write it as $(3 + i)(3 - i)$, because...

$$\begin{aligned}(3 + i)(3 - i) &= 3^2 + 3i - 3i - i^2 \\ &= 9 - (-1)^2 \\ &= 10\end{aligned}$$

Which means that

$$10 = 5 \cdot 2 = (3 + i)(3 - i)$$

So there are *two entirely different ways of factoring the same number*. This is totally wild and bizarre, and we're only doing *algebra*. Once you get into calculus with complex numbers ("complex analysis"), things get even weirder. You enter this *Alice in Wonderland* world of pure thought. That we can understand things like complex analysis (and higher mathematics in general) is a testament to the human spirit—to our ability to remove ourselves from the immediacy of the physical world.

Anyway, another cool thing about i is that when you start multiplying it by itself, a strange pattern

[1] "But how are numbers like two and three and $\sqrt{558}$ defined?," you might ask. Good question! This is beyond the scope of high school calculus—*this is why you should take more math.* If you want a good lay introduction, the first two chapters of Bertrand Russell's *Introduction to Mathematical Philosophy* (1919) are excellent.

[2] Unless you're an electrical engineer, but even then...

emerges. You've probably (at some point) seen this chart before:

$$\begin{aligned}
i &= \sqrt{-1} \\
i^2 &= (\sqrt{-1})^2 = -1 \\
i^3 &= i^2 \cdot i = -1 \cdot i = -i \\
i^4 &= i^3 \cdot i = (-i)(i) = -i^2 = 1 \\
i^5 &= i^4 \cdot i = 1 \cdot i = i \\
i^6 &= i^4 \cdot i^2 = 1 \cdot -1 = -1 \\
&\vdots \\
\text{etc.}
\end{aligned}$$

Take a moment to understand why each equality is true.

As you can see, i raised to some power goes back and forth from being a complex number to being a real number, and the pattern repeats itself every four terms. Quick—what else can you think of that repeats itself every four terms (i.e., what sort of series)? (This is when the hair on the back of your neck should start raising.)

6. (*7 points*) Show that $e^{ix} = \cos(x) + i\sin(x)$. (This is known as *Euler's formula.*) My suggestion is to write e^{ix} as a Taylor series, and then manipulate it to look like the Taylor serieses[3] of $\cos(x)$ and $i\sin(x)$. Be sure to explain every step completely, carefully, and in English!!! Don't just write the symbols! Justify your arguments! (You might want to work it out on scrap paper first and then write it nicely here.)

[3]This is a proper plural.

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7. (*3 points*) Using your result in the previous problem, prove Euler's identity—that is, show that

$$e^{i\pi} + 1 = 0$$

This is amazing. We have here, in a single equation, a relationship between the five most important numbers in math—e, i, π, 0, and 1—the three most important operations—addition, multiplication, exponentiation—and the most important relationship—equality. This is wonderous and beautiful.