Gas Laws: Balloon Animals and a Birthday Party

Calculus 12, Veritas Prep.

This is **due on Monday, December 13th**. (I'll be collecting and grading it.) Do not wait until Sunday night to start it—you won't finish. Please write up your answers on a separate sheet of paper (clearly headed with your name and the date, and with each problem clearly labelled). Please take the time to write it up *nicely*—don't hand in a paper full of scratch work. Use sentences, explain your methodology, and justify each of your steps in English and in math. Use as a model the essay I gave you a few weeks ago, "How to Write Math in Paragraph Style," by Tim Hsu[1]. I encourage you to collaborate with your classmates, but the answers you write up should be your own. Feel free to use a calculator.

1. One of the many reasons I fell in love with chemistry in high school was because of the ideal gas law:

$$PV = nRT$$

I assume you are familiar with this equation from chemistry; basically, it is an awesome (if imperfect) way to relate the pressure, volume, temperature, and number of molecules in certain gaseous systems. Temperature is measured in Kelvin, and the gas constant R is $8.314\, J \cdot K^{-1} \cdot mol^{-1} = 0.08314\, L \cdot bar \cdot K^{-1} \cdot mol^{-1} = 0.082058\, L \cdot atm \cdot mol^{-1} \cdot K^{-1}$.

If you're wondering where R comes from, it is simply another form of what is known as the *Boltzmann constant*, which relates the energy of an individual particle of matter (a microscopic quantity) to the temperature of a whole bunch of particles (a macroscopic quality). The Boltzmann constant is generally considered to be one of a handful of fundamental physical constants, and it is named (somewhat inaccurately) after one of my favorite scientists, Ludwig Boltzmann, who was so cool that he actually went insane.

Anyway, the difference between the gas constant R and the Boltzmann constant (usually written as k_b) is that the gas constant is per mole, whereas the Boltzmann constant is per particle. So $k_b = R/6.022 \cdot 10^{23} = 1.380 \cdot 10^{-23}$

2. Anyway, solve the ideal gas law for **a)** P. **b)** Now solve it for V. **c)** Now solve it for T.

3. Imagine you have a friend who loves to make flying balloon animals, and so as a birthday gift, you buy him a tank of helium. When you lug the tank over to his apartment for his birthday party, the tank weighs 25 lbs. When you bring it back the next morning, the tank is 18 grams lighter. How much helium did your friend use? Give your answer in **a)** grams, then **b)** kilograms, then **c)** moles.

4. You live, by the way, in Toronto, and your friend's birthday is on November 2nd. The temperature last November 2nd in Toronto at 11pm was 50^o F[2]. **a)** Except, Toronto is in Canada, so it would have been measured in Celsius—what's 50^oF in degrees C? And, you're a scientist, so you measure it in Kelvin. **b)** What's 50^oin Kelvin? And now onto the real question: **c)** what was the volume of helium that you let out of the tank?

5. Your friend can make all sorts of balloon-animal creations! He can make balloon zebras (with alternating black-and-white balloons); he can make balloon alligators; he can even make balloon penguins! They are all sorts of different shapes and sizes. However, after careful observation and measurement, you have determined that his average balloon animal contains about $3L$ of gas. Approximately how many balloon animals did he make?

6. Your friend's specialty, though, is making balloon Leo Tolstoys. This requires quite a a few balloons—he has a really elaborate design, complete with a Russian hat. Each balloon Tolstoy needs about $10L$ of gas to make. Assuming you insisted that he use all of the helium to make inflated Russian novelists, how many balloon Tolstoys did he make?

[1] also online at `http://www.math.sjsu.edu/~hsu/`

[2] http://www.wunderground.com/history/airport/CYTZ/2009/11/2/DailyHistory.html

7. Anyway... let's get to the calculus. Imagine you want to find how the pressure of an ideal gas changes as you change the volume (while holding the temperature and the number of moles constant). **a)** So do that (i.e., find $\frac{dP}{dV}$). **b)** Then find how the volume of an ideal gas changes as you change the number of moles but keep the pressure and temperature constant (i.e., find $\frac{dV}{dn}$). **c)** What if you change the volume, keeping the number of moles constant but not necessarily keeping the same pressure or temperature? How does the pressure change in this case?

8. Now, imagine that, based on his lung capacity and the strength of his diaphragm muscles, your friend can exhale the entire contents of his lungs (about $4.5L$) in three seconds. So imagine that he is trying to inflate the balloon-animal balloons. You already know (you figured it out back in question 2, and you've used it since) that $V = \frac{nRT}{P}$. **a)** Differentiate this equation with respect to time t (while holding T and P constant). (Time doesn't show up in this equation, so you'll need to do an implicit differentiation-style chain rule...) **b)** Find an expression for the rate at which the number of moles of helium in the balloon changes as your friend inflates it. **c)** At the instant the two-liter balloon is half-inflated, how many moles of helium are entering the balloon per second? (I know you guys love these related rates problems!)

9. After your friend makes the 10 balloon Leo Tolstoys[3], you go up to the roof of his apartment building to set them free. They float over the city and flicker under the northern lights. As they rise higher in the atmosphere, the pressure drops, and so their volumes will increase, and you'll get these giant, swollen Tolstoys floating all over Toronto. I was going to ask you to come up with an equation for $\frac{dV}{dt}$, given some knowledge about how fast they're rising and how atmospheric pressure decreases with elevation, but it gets really messy, because temperature drops with elevation, too, so I spent half an hour trying to work out an equation for the size of these balloons as a function of time, and it was doable, but a total mess. Oh well. So we'll skip the differential equation, and I'll just ask this: when the inflated Tolstoys are 30,000 feet over Toronto (where pressure is about a third of that at sea level, and temperature is maybe -50^o C), what is their volume? (Assume the balloons haven't burst. These are, like, weather-balloon balloon-animals.)

10. Let's do an actual differential equation now. With the leftover helium, your friend makes you a balloon moose. (This is Canada, after all.) It's kind of a small moose—a mooselet, really—and it only uses $1L$ of helium. You take it home, and over time, the helium slowly leaks out of the balloon, such that $\frac{dn}{dt} = -kn$, for some constant k. After two days, the moose is only half its original size. **a)** Assuming the pressure and temperature in your apartment remain constant, come up with an equation for the number of moles of helium in the moose balloon as a function of time. **b)** Then come up with an equation for the volume of the moose as a function of time. **c)** Then come up with an equation for the rate at which the volume of your moose changes.

11. Here's another one. Imagine you have a balloon, the pressure inside of which changes over time. You know the volume of the balloon is shrinking at the rate $\frac{dV}{dt} = -t^3$, and you know that initially, the pressure inside the balloon is $5\,atm$. What is the pressure inside the balloon as a function of time? (Hint: Find another way of writing $\frac{dV}{dt}$, a way which, by the chain rule, will include $\frac{dP}{dt}$. Then solve for $\frac{dP}{dt}$ and treat it as a separable differential equation.)

12. The ideal gas laws are pretty cool, but they make a couple of assumptions that cause them to break down under certain conditions. **a)** For example, what happens to the volume of an ideal gas as the pressure gets bigger and bigger (while holding temperature and number of moles constant)? (That is, solve for V and consider what happens as $P \to \infty$.) You should get that the volume goes to zero. But this doesn't really make sense—you can't just keep compressing a substance forever and make it smaller and smaller, not while keeping the individual atoms the same. Sure, you can reduce the amount of space between the individual gas molecules, but you can't really make the gas molecules themselves smaller. So the ideal gas law break down at high pressures, because it considers each

[3] This is the answer you should have gotten in 6

individual gas molecule to be infinitely small (a "point particle"). Likewise, the ideal gas law doesn't take into account intermolecular forces, which is usually not a problem, but at high pressures and temperatures—when the gas molecules are close together—this can also be a relevant quantity.

13. So, one way of improving on the ideal gas law is the *van der Waals Equation*. (You have no idea how excited I was when I first read about it in high school.) It is the following:

$$\left(P + \frac{n^2 a}{V^2}\right)(V - nb) = nRT$$

where a and b are constants that are unique for each gas and can be empirically determined. b represents, more or less, the size of each individual gas particle, and a is a measure of their intermolecular attraction.

14. **a)** Using the ideal gas law, predict the pressure that one mole of methane (CH_4) will exert on a $250mL$ container at 25^oC. **b)** Then, predict the pressure using the the van der Waals equation. (For CH_4, $a = 2.272\, L^3 \cdot atm \cdot mol^{-2}$ and $b = 0.043067\, L \cdot mol^{-1}$.) Give your answer in atmospheres. **c)** The experimentally-measured value is 77.6 atm—which equation of state gives a better result?[4]

15. Show that, as V becomes large, the van der Waals equation reduces to the ideal gas law. (This is basically just taking the limit as $V \to \infty$, but the argument is a little bit more detailed, and may feel unsatisfying... you might want to read the section in your textbook, chapter 6.7, on relative rates of growth.)

[4]Data and problem from McQuarrie & Simon, *Physical Chemistry: A Molecular Approach*, University Science Books, 1997, pp.663-4