Alternative Proofs

Calculus 11, Veritas Prep.

Probably our most basic derivative rule is the additive rule—the theorem that says that $\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$. Sensibly enough, we proved this by starting with Fermat's difference quotient. The FDQ is our fundamental definition of a derivative, and it's not too hard to plug $f(x)+g(x)$ into it, rearrange things a bit, and get $f'(x)+g'(x)$. But—as we discovered during one 11th grade class—we can prove it entirely differently. We can prove it just using our knowledge of the derivatives of logs and exponentials, and the chain rule. The idea is that $f(x)+g(x)$ is equal to $\ln\left(e^{f(x)+g(x)}\right)$, by basic properties of logs, so if I want to take the derivative of the former, I can just take the derivative of the latter:

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}\left[\ln(e^{f(x)+g(x)})\right]$$

but I can use properties of exponents to rewrite this as:

$$= \frac{d}{dx}\left[\ln\left(e^{f(x)}e^{g(x)}\right)\right]$$

so I just do the derivative of a log:

$$= \frac{1}{e^{f(x)}e^{g(x)}} \cdot \frac{d}{dx}\left[e^{f(x)}e^{g(x)}\right]$$

and the chain rule "derivative of the inside" part I can evaluate using the product rule:

$$= \frac{1}{e^{f(x)}e^{g(x)}} \cdot \left(f'(x)e^{f(x)}e^{g(x)} + g'(x)e^{g(x)}e^{f(x)}\right)$$

but then I can factor out the $f'(x)+g'(x)$:

$$= \frac{1}{e^{f(x)}e^{g(x)}} \cdot \left(f'(x)+g'(x)\right) \cdot \left(e^{f(x)}e^{g(x)}\right)$$

and then everything else cancels:

$$= f'(x)+g'(x)$$

Woah! So we've used a couple of derivative laws in here—we've used the laws about the derivative of an exponential, the derivative of a log, and the chain rule. The derivative of a log and the chain rule we can both prove using only Fermat's difference quotient (and the derivative of a log requires only the derivative of an exponential). So from just two applications of the FDQ, we can get all these other things! We don't have to keep using the FDQ! Compare to the two ways of proving the quotient rule: we proved it first using the FDQ, and then using a combination of the product rule and the chain rule. We can base all of our shortcuts on the FDQ, or we can try to base our shortcuts on other shortcuts based on shortcuts based on the FDQ.

Problem

Without using Fermat's difference quotient, prove the product rule. Then prove the constant multiple rule. For some extra credit, write the proofs up nicely and turn them in before break.

Another fun thing to do would be to draw a "dependency chart" for our various theorems: can you visually represent which derivative rules are based on which other derivative rules (like a family tree, but a family in which Fermat's difference quotient is the ultimate patriarch and everyone else is related through incestuous logic that spans and skips generations)?